EPJ Web of Conferences **22**, 00005 (2012)
DOI: 10.1051/epjconf/20122200005

An introduction to the linear representations of finite groups

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Abstract. A few elements of the formalism of finite group representations are recalled. As to avoid a too mathematically oriented approach the discussed items are limited to the most essential aspects of the linear and matrix representations of standard use in chemistry and physics.

1. INTRODUCTION

Symmetry is ubiquitous in nature and of an extremely wide variety. It may be discrete, such as the space inversion, the time inversion, the crystal isometries, ..., or continuous, such as the euclidean isometries, the galilean invariance, the gauge invariances, ... It may be obvious, generally when it is of geometric nature. It may be hidden, often when it is of dynamical origin, then revealing itself indirectly.[1] It may be more or less blurred, typically as perceived in the complex systems, botanical, biological, ... It may be spontaneously broken, in which instance it becomes the source of a number of non-trivial phenomena, including the phase transitions, the bifurcations in the non linear processes, ..., that gives rise to a wealth of structuration. It suffices for illustration to evoke the uncountable physical phases of matter, for instance the crystalline and mesomorphic forms or else the magnetic orders among the most familiar categories, not to mention the dynamic self-organization, the pattern formation, ... found out in the other fields. Symmetry in fact scarcely is lowered uniformly so that the broken phase spatially builds up from different states, transforming into one another by the lost components of the symmetry, and thus is non uniform and displays defects. These in turn might interact or cross, possibly non commutatively, to organize themselves or generate further novel textures.

Symmetry gets materialized through a set of transformations of the properties of a system, which, endowed with the canonical composition law for functions, forms a group whatever the instance. Accordingly, the adequate framework within which to deal with symmetry is that of the group theory, including its ramifications into the representation theory to account for the nature of the invariances of the physical properties, the differential geometry, in particular the Morse theory, to investigate the extrema of the invariant functions of the physical properties and thus to get insights into the symmetry breaking phenomena, the algebraic topology, more specifically the homotopy theory, to feature the topological stability of defects and the formation of textures, ... It is clear that this is too vast a field to

[1] A case in point is provided by the bound states of the non relativistic isolated hydrogen atom, which displays spectral degeneracies with respect to the principal n and orbital l quantum numbers. Whereas the l-degeneracy is an evident outcome of the symmetry group SO(3) of the rotations in the 3-dimensional space $\mathbb{R}^3$, the n-degeneracy is specific to the Kepler potentials, decreasing as the inverse of the radial distance, and emanates from the dynamical symmetry group SO(4). Considering the scattering states of the continuum in the spectrum, this metamorphoses itself into the dynamical symmetry group SO(3,1). In other words, using a more intuitive picture, the electron dynamics in a $1/r$ potential is equivalent to that of a free particle in the 4-dimensional space $\mathbb{R}^4$, on a sphere $\mathcal{S}_3$ if it is bounded and on a double-sheeted hyperboloid $\mathcal{H}_3$ if it is scattered. Another feature of the electron spectrum is the equal spacing of the energy levels when multiplied by $-n^3$, which suggests duality and originates from the De Sitter spectrum generating symmetry group SO(4,1). Attempts to express the hamiltonian in terms of operators that close under commutation lead to anticipate that the largest spectrum generating symmetry group of the hydrogen atom might be the conformal group SO(4,2).

describe in a few pages. The ambition of these notes is strongly limited. It is to focus on the mathematics of the linear representation of finite groups. After short recalls of basic concepts, questions of reduction and irreducibility are discussed. Next, character theory is succinctly explored. Complete reducibility of the linear representations of finite groups, the relevance and usefulness of the Schur's Lemmas, complete invariant nature of the characters with respect to intertwining and character completeness over class functions are emphasized. Construction of induced linear representations will be approached and search methods of irreducible representations will be mentioned only briefly. Of course, the discussed items are far from providing even the rough idea of all the richness of the group representations. A number of their facets are only alluded to or merely ignored, for instance concerned with the multi-valued spinor representations, the projective representations, ..., not to mention the linear representations of continuous groups or else the non linear group actions. An extremely wide literature exists on these topics, quite often purely mathematical, including textbooks or reviews to start with. See for instance [1–5].

2. BASIC CONCEPTS

A **representation of a group G on a mathematical object X** designates an **homomorphism** $\rho : \mathrm{G} \rightarrow \mathrm{Aut}(\mathrm{X})$ from the group G to the automorphism group Aut(X) of the object X:

$$\rho(gh) = \rho(g) \circ \rho(h) \quad \forall g \in \mathrm{G}\ \forall h \in \mathrm{G} \tag{2.1}$$

G may be any group, finite or infinite, possibly topological in which case it may be (locally) compact or non compact, n-connected, ... X may be any set endowed with a mathematical structure, for instance a topological space, a differentiable manifold, a module over a ring, ... Aut(X) is the group formed by the set of the bijective functions $\mathfrak{f} : \mathrm{X} \rightarrow \mathrm{X}$ that preserve the mathematical structure of X, endowed with the canonical composition law $\circ$ for the functions.

If X is a **vector space** V over a scalar field $\mathbb{K}$ then Aut(V) is the group $\mathrm{GL}(\mathrm{V}, \mathbb{K})$ of the **invertible linear operators** on V:

$$\rho(g)(a\ \vec{u} + b\ \vec{v}) = a\ \rho(g)(\vec{u}) + b\ \rho(g)(\vec{v}) \quad \forall g \in \mathrm{G}\ \forall (a, b) \in \mathbb{K}^2\ \forall (\vec{u}, \vec{v}) \in \mathrm{V}^2 \tag{2.2}$$

In this case ρ is particularized by naming it a **linear representation**. V is the **representation space**. It is customary to call **dimension of the representation** the dimension d of V. Only the linear representations of the **finite groups** G on the vector spaces V over the field $\mathbb{C}$ of the **complex numbers**[2] are discussed in this manuscript, unless otherwise explicitly stated.

With every linear representation $\rho : \mathrm{G} \rightarrow \mathrm{GL}(\mathrm{V}, \mathbb{K})$ is associated its **kernel** $ker(\rho)$ and its **image** $im(\rho)$, given as

$$ker(\rho) = \{g \in \mathrm{G} \mid \rho(g) = 1_{\mathrm{V}}\} \quad \text{and} \quad im(\rho) = \{\rho(g) \mid g \in \mathrm{G}\} \tag{2.3}$$

where $1_{\mathrm{V}} \in \mathrm{GL}(\mathrm{V}, \mathbb{K})$ is the identity operator on the representation space V. If $\rho(g) = \rho(h)$ then $gh^{-1} \in ker(\rho)$. It follows that ρ is injective if and only if (iff) $ker(\rho) = \{e\}$, where e is the unit element of G. ρ by definition is surjective iff $im(\rho) = \mathrm{GL}(\mathrm{V}, \mathbb{K})$. If $(g, h) \in ker(\rho)^2$ then $\rho(gh^{-1}) = 1_{\mathrm{V}}$, namely $gh^{-1} \in ker(\rho)$, which implies that $ker(\rho)$ is a subgroup of G. It similarly is shown that $im(\rho)$ is a subgroup of $\mathrm{GL}(\mathrm{V}, \mathbb{K})$. If $g \in \mathrm{G}$ and $h \in ker(\rho)$ then $\rho(ghg^{-1}) = 1_{\mathrm{V}}$, namely $ghg^{-1} \in ker(\rho)$, which

[2] An evident motivation to restrict the scalar field $\mathbb{K}$ to the field $\mathbb{C}$ of the complex numbers is of course that physics suggests it as natural, together with the field $\mathbb{R}$ of the real numbers. A mathematical motivation is that this avoids unnecessarily cautioning against a number of algebraic stuffs, because $\mathbb{C}$ has **zero characteristic**, similarly as $\mathbb{R}$, and is **algebraically closed**, in contrast to $\mathbb{R}$. These two mathematical properties are relevant to certain aspects of crucial theorems, such as the Complete Reducibility Theorem, the Schur's Lemma, ... The characteristic $char(\mathbb{K})$ of a field $\mathbb{K}$ is the positive integer $n_{\mathbb{K}}$ the multiples of which makes up the kernel $n_{\mathbb{K}}\mathbb{Z}$ of the homomorphism $\varphi : m \mapsto 1_{\mathbb{K}} + \ldots + 1_{\mathbb{K}} = m \cdot 1_{\mathbb{K}}$ of the additive group of the integer numbers $\mathbb{Z}$ to the additive group of the $\mathbb{K}$-scalars. $char(\mathbb{K})$ by convention is set to zero whenever $n_{\mathbb{K}}$ is not finite. A field is algebraically closed if for every polynomial $\mathrm{P}(z)$ of one variable and coefficients in this field, $\exists z_0$ s.t. $\mathrm{P}(z_0) = 0$.

means that $ker(\rho)$ is a normal subgroup of G,

$$ker(\rho) \trianglelefteq \mathrm{G} \tag{2.4}$$

A quotient group $\mathrm{G}/ker(\rho)$ is canonically defined by endowing the set of the left cosets of $ker(\rho)$ with the internal law of composition $\mu : (g\, ker(\rho), h\, ker(\rho)) \mapsto (gh)\, ker(\rho)$. It almost is obvious that $\mathrm{G}/ker(\rho)$ is isomorphic to $im(\rho)$,

$$\mathrm{G}/ker(\rho) \cong im(\rho) \tag{2.5}$$

All these properties actually are generic to the kernel and image of any homomorphism of groups.

A linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{K})$ is **isomorphic** by definition to a linear representation $\xi : \mathrm{G} \to \mathrm{GL}(\mathrm{W}, \mathbb{K})$ if there exists an isomorphism $\sigma : \mathrm{V} \to \mathrm{W}$, which is equivariant:

$$\xi(g) \circ \sigma = \sigma \circ \rho(g) \quad \forall g \in \mathrm{G} \tag{2.6}$$

σ is called an **intertwining operator** or a G-linear map. A standard notation for isomorphic representations is $\boldsymbol{\rho \sim \xi}$. The isomorphism of representations is reflexive, symmetric and transitive, so is an **equivalence relation**, which gathers the linear representations into equivalence classes.

Any vector space V possesses a dual $\mathrm{V}^\#$, which canonically is built up by endowing the set of linear forms on V with pointwise addition $(\vec{u}^\# + \vec{v}^\#)(\vec{w}) = \vec{u}^\#(\vec{w}) + \vec{v}^\#(\vec{w})$ and pointwise scalar multiplication $(\lambda\, \vec{v}^\#)(\vec{w}) = \lambda\, \vec{v}^\#(\vec{w})$, where $\lambda \in \mathbb{C},\ \vec{w} \in \mathrm{V},\ \vec{u}^\# \in \mathrm{V}^\#$ and $\vec{v}^\# \in \mathrm{V}^\#$.[3] Let $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ be a linear representation. An application $\rho^\# : \mathrm{G} \to \mathrm{GL}(\mathrm{V}^\#, \mathbb{C}),\ g \mapsto \rho^\#(g)$ such that

$$\left(\rho^\#(g)(\vec{v}^\#)\right)(\rho(g)(\vec{u})) = \vec{v}^\#(\vec{u}) \quad \forall g \in \mathrm{G}, \quad \forall \vec{u} \in \mathrm{V}, \quad \forall \vec{v}^\# \in \mathrm{V}^\# \tag{2.7}$$

can be defined. With $\vec{w} = \rho(g)(\vec{u})$, this is rewritten $\left(\rho^\#(g)(\vec{v}^\#)\right)(\vec{w}) = \vec{v}^\#\left(\rho(g)^{-1}(\vec{w})\right)$. In other words $\rho^\#(g)(\vec{v}^\#) = \vec{v}^\# \circ \rho(g^{-1})$, which makes up another equivalent defining relation and clearly shows that $\rho^\#(g)$ does exist and is unique thanks to the existence and unicity of $\rho(g^{-1})\ \forall g \in \mathrm{G}$. Moreover, $\rho^\#(gh)(\vec{v}^\#) = \vec{v}^\# \circ \rho((gh)^{-1}) = \vec{v}^\# \circ \rho(h^{-1}) \circ \rho(g^{-1}) = \rho^\#(g)(\vec{v}^\# \circ \rho(h^{-1})) = \rho^\#(g)(\rho^\#(h)(\vec{v}^\#)) = (\rho^\#(g) \circ \rho^\#(h))(\vec{v}^\#),\ \forall \vec{v}^\# \in \mathrm{V}^\#,\ \forall (g, h) \in \mathrm{G}^2$, which demonstrates that $\rho^\#$ is a group homomorphism. $\rho^\#$ is the **dual representation** of ρ. All the theorems established for ρ are valid for $\rho^\#$, and conversely, by mere structure transport.

2.1 Canonical examples

Automorphism groups $\mathrm{GL}(\mathrm{V}_{d=1}, \mathbb{C})$ of 1-dimensional vector spaces $\mathrm{V}_{d=1}$ are isomorphic to the multiplicative group $\mathbb{C}^\star$ of non null complex numbers, insofar as every invertible linear operator η on $\mathrm{V}_{d=1}$ is equivalent to the multiplication by a same non null scalar: if $\hat{e}$ is the basis vector in $\mathrm{V}_{d=1}$ then $\exists! a \in \mathbb{C}^\star$ s.t. $\eta(\hat{e}) = a\hat{e}$ ergo $\forall \vec{v} \in \mathrm{V}_{d=1}\ \eta(\vec{v}) = \eta(v\hat{e}) = v\eta(\hat{e}) = va\hat{e} = av\hat{e} = a\vec{v}$. **Any homomorphism $\boldsymbol{\rho : \mathrm{G} \to \mathbb{C}^\star}$** thus makes up **a linear representation of dimension $\boldsymbol{d = 1}$** of the group G. An evocative example is $\delta : \mathrm{GL}(\mathrm{d}, \mathbb{C}) \to \mathbb{C}^\star,\ M \mapsto \mathrm{Det}(M)$, where $\mathrm{GL}(\mathrm{d}, \mathbb{C})$ designates the group of $d \times d$ non singular matrices with entries in $\mathbb{C}$ and $\mathrm{Det}(M)$ the determinant of a matrix M. $ker(\delta) = \mathrm{SL}(d, \mathbb{C})$ consists in the $d \times d$ matrices with determinant 1, which thus is a normal subgroup of $\mathrm{GL}(\mathrm{d}, \mathbb{C})$. Since $im(\delta) = \mathbb{C}^\star$ we have $\mathrm{GL}(\mathrm{d}, \mathbb{C})/\mathrm{SL}(d, \mathbb{C}) \cong \mathbb{C}^\star$. $\mathrm{GL}(\mathrm{d}, \mathbb{C})$ is called the general linear group of order d over $\mathbb{C}$ and $\mathrm{SL}(d, \mathbb{C})$ the special linear group of order d over $\mathbb{C}$. It

[3] A linear form by definition is an application $\vec{v}^\# : \mathrm{V} \to \mathbb{C}$ from a vector space V to its scalar field $\mathbb{C}$ such that $\vec{v}^\#(a\, \vec{r} + b\, \vec{s}) = a\, \vec{v}^\#(\vec{r}) + b\, \vec{v}^\#(\vec{s}),\ \forall (a, b) \in \mathbb{C}^2,\ \forall (\vec{r}, \vec{s}) \in \mathrm{V}^2$. It also is called a one-form, a linear functional, a co-vector, a contravariant vector when the elements of V are called covariant vectors, ... This merely emphasizes the wealth of context within which the concept might be in use, such as differential geometry, measure theory, multilinear algebra, ... If V has the finite dimension d then $\mathrm{V}^\#$ has the same dimension d. A basis $\{\hat{e_i}^\#\}_{i=1,\ldots,d}$ in $\mathrm{V}^\#$ is twinned in fact to any selected basis $\{\hat{e_i}\}_{i=1,\ldots,d}$ in V such that $\hat{e_i}^\#(\hat{e_j}) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ iff $i = j$ and $\delta_{ij} = 0$ otherwise). When V is infinite-dimensional the same construction does not end up with a basis. It leads to a family of linearly independent vectors that is not spanning. The linear forms on a finite-dimensional normed space V are bounded and therefore are continuous.

is clear that $g^m \in \mathrm{G} \Rightarrow g^{m+1} \in \mathrm{G}\ \forall m \in \mathbb{N}$ whence, by increasing to infinity, m crosses integers p for which there exists strictly positive integers $q < p$ such that $g^p = g^q$ or else $g^{p-q} = e$, unless G is infinite. In other words, whenever the group **G is finite** each of its element g is of finite order $n_g = \text{Minimum}\,[r \in \mathbb{N}^\star \mid g^r = e]$. Obviously, $\rho(g)^{n_g} = \rho(g^{n_g}) = \rho(e) = 1$, which means that **$\rho(g)$ is an n_g-th root of 1**, the multiplicative unit of $\mathbb{C}^\star$. Now whatever the group G, finite or not, in the event that one has

$$\rho(g) = 1 \quad \forall g \in \mathrm{G} \tag{2.8}$$

ρ is called the **trivial representation** of the group G. Its significance is to reveal the full invariance of a physical property with respect to the symmetries abstracted by the elements of the group G.

Indexing with the elements x of a finite set X the basis vectors $\hat{e}_x$ of a vector space V and associating each element g of a finite group G with the invertible linear operator $\rho_\mathrm{X}(g)$ on V that sends $\hat{e}_x$ to $\hat{e}_{\pi(g)(x)}$, where $\pi : \mathrm{G} \to \mathcal{P}_\mathrm{X}$ is an homomorphism of the group G into the group $\mathcal{P}_\mathrm{X}$ of the permutations of X, generates a linear representation ρ_X, which is called the permutation representation of the group G associated with the set X. Note that the group homomorphism $\pi : \mathrm{G} \to \mathcal{P}_\mathrm{X}$ defines a representation of the group G on the set X. It is the usage in that case to state that the group G acts on the set X or else that X is a **G-set**. In the specific instance where the set X contains the same number n_G of elements as the group G the permutation representation is isomorphic to the so-called **regular representation** ρ_G of the group G. One conventionally defines ρ_G by indexing the basis vectors of the vector space V with the elements h of the group G, more concisely as $\hat{e}_h$ where $h \in \mathrm{G}$, and by associating each element g of the group G with the invertible linear operator $\rho_\mathrm{G}(g)$ on V that transforms the basis vectors, thus G-indexed, according to the formula

$$\rho_\mathrm{G}(g)(\hat{e}_h) = \hat{e}_{gh} \quad \forall g \in \mathrm{G}\ \forall h \in \mathrm{G} \tag{2.9}$$

The regular representation ρ_G is particularized because containing each irreducible representation ρ_i of the group G with a repetition factor equal to its dimension d_i. The dimension of ρ_G is the order n_G of the group G. The set $\{\rho_\mathrm{G}(g)(\hat{e}_e) \mid g \in \mathrm{G}\}$, engendered from the single vector $\hat{e}_e$ indexed with the unit element e of the group G, forms a basis of the representation space V. Conversely, given a linear representation $\xi : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$, if there exist a vector $\vec{v}$ in the representation space V such that the set $\{\xi(g)(\vec{v}) \mid g \in \mathrm{G}\}$ forms a basis of V then ξ necessarily is isomorphic to ρ_G. Consider indeed the isomorphism $\sigma : \mathrm{V} \to \mathrm{V}$ defined by setting $\sigma(\hat{e}_h) = \xi(h)\vec{v}$. Since ξ is an homomorphism, $\forall (g,h) \in \mathrm{G}^2,\ \xi(g)\,(\xi(h)(\vec{v})) = \xi(gh)(\vec{v})$, but, by definition of σ, $\xi(h)(\vec{v}) = \sigma(\hat{e}_h)$ and $\xi(gh)(\vec{v}) = \sigma(\hat{e}_{gh})$ so that $\xi(g)\,(\sigma(\hat{e}_h)) = \sigma(\rho_\mathrm{G}(g)(\hat{e}_h))$, which implies that $\xi(g) \circ \sigma = \sigma \circ \rho_\mathrm{G}(g)\ \forall g \in \mathrm{G}$, namely that σ is equivariant, whence $\xi \sim \rho_\mathrm{G}$.

2.2 Matrix representations

Let V be a vector space with dimension d over the field $\mathbb{C}$. Any element $\rho(g)$ of the group GL(V,$\mathbb{C}$) of the invertible linear operators on V is fully determined from the images $\rho(g)(\hat{e}_m)$ of the basis vectors $\hat{e}_m$ $(m = 1, \ldots, d)$ selected in V. Indeed, $\forall \vec{v} \in \mathrm{V}\ \exists!\ (x_1,\ \ldots,\ x_d) \in \mathbb{C}^d : \vec{v} = \sum_m x_m \hat{e}_m$ so that $\rho(g)(\vec{v}) = \sum_m x_m \rho(g)(\hat{e}_m)$. Now, $\rho(g)(\hat{e}_m)$ is a vector of V. Accordingly, $\exists!\ (\Gamma(g)_{1m},\ \ldots,\ \Gamma(g)_{dm}) \in \mathbb{C}^d$ such that

$$\rho(g)(\hat{e}_m) = \sum_n \hat{e}_n\ \Gamma(g)_{nm} \tag{2.10}$$

If $\rho(g)(\vec{v}) = \sum_n y_n\ \hat{e}_n$ then $y_n = \sum_m \Gamma(g)_{nm}\ x_m$. The d^2 complex coefficients $\Gamma(g)_{nm}$ make up the entries of a $d \times d$ invertible matrix $\Gamma(g)$, called the **matrix representative** of the linear operator $\rho(g)$. Assume that $\rho(g)$ generically symbolizes the image of an element g of a group G by a linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$, so that $\forall\ g, h \in \mathrm{G},\ \ \rho(gh) = \rho(g) \circ \rho(h)$. It follows from the equation (2.10) that $\rho(gh)(\hat{e}_m) = \sum_n \hat{e}_n\ \Gamma(gh)_{nm}$ and $\rho(g) \circ \rho(h)(\hat{e}_m) = \rho(g)(\sum_s \hat{e}_s\ \Gamma(h)_{sm}) = \sum_n \hat{e}_n\ [\sum_s \Gamma(g)_{ns}\Gamma(h)_{sm}]$. Accordingly,

$$\Gamma(gh) = \Gamma(g)\Gamma(h) \quad \forall g \in \mathrm{G}\ \forall h \in \mathrm{G} \tag{2.11}$$

This means that the mapping $\Gamma : \mathrm{G} \to \mathrm{GL(d,\mathbb{C})}$ of the group G to the group $\mathrm{GL(d,\mathbb{C})}$ of $d \times d$ invertible matrices with entries in $\mathbb{C}$, which to each element g in G associates the matrix representative $\Gamma(g)$ of the linear operator $\rho(g)$ with respect to the selected basis $\{\hat{e}\}_{m=1,\dots,d}$, defines a group homomorphism. This is called a **matrix representation** of the group G.

The selection of another basis $\{\hat{f}\}_{n=1,\dots,d}$ would have led to other matrix representatives $\Lambda(g)$, giving rise to another matrix representation $\Lambda : \mathrm{G} \to \mathrm{GL(d,\mathbb{C})}$. Associated with the same linear representation ρ and merely emerging from the selection of two different bases in the representation space V, the matrix representations Γ and Λ are said **similar** or **equivalent**. If S is the invertible matrix associated with the basis change $\{\hat{e}\}_{m=1,\dots,d} \to \{\hat{f}\}_{n=1,\dots,d}$, which often is called a similarity transformation, then[4]

$$\Lambda(g) = S\,\Gamma(g)\,S^{-1} \quad \forall g \in \mathrm{G} \tag{2.12}$$

Λ and Γ are said **intertwined** with S. Conversely, any two finite dimensional matrix representations of a finite group intertwined with an invertible matrix are similar. As with the linear representations, a standard notation for two equivalent matrix representations is $\boldsymbol{\Gamma \sim \Delta}$. Now, $\Lambda(g)$ could have been interpreted also as the matrix representative with respect to the initial basis vectors $\hat{e}_m$ ($m = 1, \dots, d$) of a linear operator $\xi(g)$ associated with another linear representation $\xi : \mathrm{G} \to \mathrm{GL(V,\mathbb{C})}$. The equation (2.12) then would mean that there exists an automorphism σ of V which is equivariant: $\xi(g) = \sigma \circ \rho(g) \circ \sigma^{-1}$, $\forall g \in \mathrm{G}$ so that $\rho \sim \xi$. Conversely, any automorphism σ of V corresponds to a change of bases. Accordingly, the isomorphism of linear representations and that of matrix representations describe the same equivalence.

As from every matrix $\mathcal{M}$ with entries $\mathcal{M}_{ij}$ in $\mathbb{C}$ is built the complex conjugate $\mathcal{M}^\star$ with the entries $(\mathcal{M}^\star)_{ij} = (\mathcal{M}_{ij})^\star$, the transpose ${}^t\mathcal{M}$, by column-row interchange, with the entries $({}^t\mathcal{M})_{ij} = \mathcal{M}_{ji}$ and the adjoint $\mathcal{M}^\dagger = ({}^t\mathcal{M})^\star$ with the entries $(\mathcal{M}^\dagger)_{ij} = (\mathcal{M}_{ji})^\star$. Given a matrix representation $\Gamma : \mathrm{G} \to \mathrm{GL(d,\mathbb{C})}$, by associating each element g of the group G with the complex conjugate $\Gamma^\star(g)$, the transpose ${}^t\Gamma(g)$ and the adjoint $\Gamma^\dagger(g)$ of $\Gamma(g)$ one respectively defines the conjugate $\Gamma^\star$, the transpose ${}^t\Gamma$ and the adjoint $\Gamma^\dagger$ of the matrix representation Γ.

2.3 Direct sums

Let $\rho : \mathrm{G} \to \mathrm{GL(V,\mathbb{C})}$ be a linear representation. A **proper subspace** V_1 of the representation space V by definition is **stable** or **invariant** under the group G iff

$$\forall \vec{v}_1 \in \mathrm{V}\ \vec{v}_1 \in \mathrm{V}_1 \Rightarrow \rho(g)(\vec{v}_1) \in \mathrm{V}_1 \quad \forall g \in \mathrm{G} \tag{2.13}$$

or, in terms of subsets, $\rho(g)\mathrm{V}_1 \subseteq \mathrm{V}_1$, $\forall g \in \mathrm{G}$. A subspace V_1 of V is proper iff it is distinct from V and the zero-dimensional vector space $\{\vec{0}\}$. V and $\{\vec{0}\}$ are trivially stable under any group G. The restriction $\rho_{\mathrm{V}_1}(g)$ of $\rho(g)$ to V_1 determines an automorphism of V_1 and follows the group homomorphism rule $\rho_{\mathrm{V}_1}(gh) = \rho_{\mathrm{V}_1}(g) \circ \rho_{\mathrm{V}_1}(h)\ \forall (g, h) \in \mathrm{G}$, which means that the application

$$\rho_{\mathrm{V}_1} : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_1, \mathbb{C}),\ g \mapsto \rho_{\mathrm{V}_1}(g) \text{ s.t. } \rho_{\mathrm{V}_1}(g)(\vec{v}_1) = \rho(g)(\vec{v}_1)\ \forall \vec{v}_1 \in \mathrm{V}_1 \tag{2.14}$$

is a linear representation of the group G on the vector space V_1, which is called a **subrepresentation** of ρ.

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[4] $\rho(g)(\hat{f}_m) = \sum_n \hat{f}_n\ \Lambda(g)_{nm} = \rho(g)(\sum_r \hat{e}_r\ S^{-1}_{rm}) = \sum_r \rho(g)(\hat{e}_r)\ S^{-1}_{rm} = \sum_r \left(\sum_s \hat{e}_s\,\Gamma(g)_{sr}\right)\ S^{-1}_{rm} = \sum_r(\sum_s(\sum_n \hat{f}_n\ S_{ns}) \times \Gamma(g)_{sr})\,S^{-1}_{rm} = \sum_n \hat{f}_n\ \left(\sum_s \sum_r S_{ns}\Gamma(g)_{sr}S^{-1}_{rm}\right) = \sum_n \hat{f}_n\ (S\,\Gamma(g)S^{-1})_{nm}$.

Select a basis $\{\hat{e}_m\}$ in V_1 and extend it to a basis $\{\hat{e}_m\} \cup \{\hat{f}_n\}$ in V, which always is possible whenever V is finite-dimensional or otherwise once the axiom of choice is allowed.[5] A subspace ${V_2}^f$ of V is linearly spanned by the set of vectors $\hat{f}_n$. It is called a **complement** of the subspace V_1 in the vector space V, because any vector $\vec{v}$ in V writes uniquely as $\vec{v} = \vec{v}_1 + \vec{v}_2$ with $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in {V_2}^f$. It is observed that $V_1 \cap {V_2}^f = \{\vec{0}\}$. If V is finite-dimensional with dimension d and the dimension of V_1 is d_1 then the dimension of ${V_2}^f$ is $d_2 = d - d_1$. If, conversely, a finite-dimensional vector space V contains two subspaces, V_1 with dimension d_1 and V_2 with dimension d_2, such that $V_1 \cap V_2 = \{\vec{0}\}$ and $d = d_1 + d_2$ is the dimension of V then every $\vec{v}$ in V writes uniquely as $\vec{v} = \vec{v}_1 + \vec{v}_2$ with $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$. It may be emphasized that a complement of a proper subspace is a proper subspace and that V and $\{\vec{0}\}$ are the complements of each other in V. One symbolically formulate the fact that two proper subspaces V_1 and V_2 of a vector space V are the complements of each other in V as

$$V = V_1 \oplus V_2 \tag{2.15}$$

In the event that not only the proper subspace V_1 but also the selected complement V_2 in V is stable under the group G, the restriction $\rho_{V_2} : G \to GL(V_2, \mathbb{C})$ of ρ to the representation space V_2 makes up another subrepresentation of ρ. Importantly, $\forall g \in G\ \forall \vec{v} \in V,\ \rho(g)(\vec{v})$ is fully and uniquely determined by the sum $\rho_{V_1}(g)(\vec{v}_1) + \rho_{V_2}(g)(\vec{v}_2)$ with $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$. In addition, $\vec{v}_i \in V_i \Rightarrow \rho_i(g)(\vec{v}_i) \in V_i$ and $\vec{v}_j \in V_{j \neq i} \Rightarrow \rho_{V_i}(g)(\vec{v}_j) = 0\ (i, j = 1, 2)$. It is customary to transcribe these properties by symbolically equating ρ to the direct sum of ρ_{V_1} and ρ_{V_2}:

$$\rho = \rho_{V_1} \oplus \rho_{V_2} \tag{2.16}$$

With respect to the basis $\{\hat{e}_m\} \cup \{\hat{e}_n\}$, built by union of the basis $\{\hat{e}_m\}$ in V_1 and the basis $\{\hat{e}_n\}$ in V_2, the matrix representatives $\Gamma(g)$ of the linear operators $\rho(g)$ on V write in the **block diagonal** form

$$\Gamma(g) = \Gamma^1(g) \oplus \Gamma^2(g) \equiv \begin{pmatrix} \Gamma^1(g) & 0 \\ 0 & \Gamma^2(g) \end{pmatrix} \quad \forall g \in G \tag{2.17}$$

namely as the direct sum $\Gamma^1(g) \oplus \Gamma^2(g)$ of the matrix representatives $\Gamma^1(g)$ of the linear operators $\rho_{V_1}(g)$ on V_1 with respect to the basis $\{\hat{e}_m\}$ and of the matrix representatives $\Gamma^2(g)$ of the linear operators $\rho_{V_2}(g)$ on V_2 with respect to the basis $\{\hat{e}_n\}$. Again, now to implicitly recall the block-diagonal structure of the matrix representatives $\Gamma(g)$, it is the convention to symbolically write

$$\Gamma = \Gamma^1 \oplus \Gamma^2 \tag{2.18}$$

and, subsequently, to state that the matrix representation Γ is the **direct sum** of the sub-matrix representations Γ^1 and Γ^2.

As an illustration, let $\rho_G : G \to GL(V, \mathbb{C})$ be the regular representation of a group G on the vector space V with basis $\{\hat{e}_g\}_{g \in G}$ and let V_1 be the one-dimensional subspace of V consisting in the scalar multiples of the vector $\sum_{g \in G} \hat{e}_g$. V_1 evidently is stable under G: $\forall \vec{v}_1 \in V_1\ \exists!\ a \in \mathbb{C} : \vec{v}_1 = a \sum_{h \in G} \hat{e}_h$ so that $\forall g \in G\ \rho_G(\vec{v}_1) = a \sum_{h \in G} \hat{e}_{gh} = \vec{v}_1 \in V_1$. Let V_2 be the subspace of V spanned by the $n_G - 1$ vectors $(\hat{e}_h - \hat{e}_e)_{h \in (G - \{e\})}$, where e is the unit element of G and n_G is the order of G. It easily is shown that the dimension of the subspace V_2 is $n_G - 1$ and that $V_1 \cap V_2 = \{\vec{0}\}$, by noticing that if $a \sum_{h \in G} \hat{e}_h = \sum_{h \in G, h \neq e} b_h (\hat{e}_h - \hat{e}_e)$ then $\sum_{h \in G, h \neq e} (b_h - a) \hat{e}_h - (\sum_{h \in G, h \neq e} b_h + a) \hat{e}_e = 0$, ergo $a = 0$ and $b_{h \neq e} = 0\ \forall h \in G$. Next, by applying the linear operators $\rho_G(g)$ on the $n_G - 1$ vectors $(\hat{e}_h - \hat{e}_e)_{h \in (G - \{e\})}$ as $\rho_G(g)(\hat{e}_{h \neq e} - \hat{e}_e) = \hat{e}_{gh} - \hat{e}_g = (\hat{e}_{gh} - \hat{e}_e) - (\hat{e}_g - \hat{e}_e)$, it is straightforwardly inferred that the subspace V_2 is stable under G. Accordingly, the subspaces V_1 and V_2 thus constructed are effectively complement

5 The axiom of choice is not universally accepted because it leads to strange theorems, the most famous being the Banach-Tarski paradoxical decomposition. Ignoring it however also leads to disasters, for instance a vector space may have no basis or may have bases with different cardinalities. As to cure some of the inconveniences, in particular the existence of non-measurable sets of reals, the axiom of determinacy was put forward in replacement, but this still might not be all satisfactory. Under this axiom every subset of the set of reals $\mathbb{R}$ is Lebesgue-measurable, but, for instance, $\mathbb{R}$ as a vector space over the set of rationals $\mathbb{Q}$ has no basis.

of each other and invariant under G so that ρ_G can be put into the direct sum of the subrepresentations built over these proper subspaces. Another choice of complement could have been made with the $n_G - 1$ vectors $\hat{e}_{h\in(G-\{e\})}$, but this is not stable under G. It suffices to observe that $\rho_G(g)(\hat{e}_{g^{-1}}) = \hat{e}_e$ does not belong to this complement.

2.4 Maschke's theorem

A convenient tool to handle the direct sums of proper subspaces is the **projection operator**. It is recalled that given the decomposition $V = V_1 \oplus {V_2}^f$, every vector $\vec{v}$ in V by definition writes uniquely as $\vec{v} = \vec{v}_1 + \vec{v}_2$ with $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in {V_2}^f$. The linear operator π_f that sends every vector $\vec{v}$ in V onto its component $\vec{v}_1$ in V_1 defines the projection operator of V onto V_1 along ${V_2}^f$. It is clear that $\pi_f \circ \pi_f = \pi_f$. It also is almost obvious that the image of π_f is $im(\pi_f) = \{\pi_f(\vec{v}) \mid \vec{v} \in V\} = V_1$, the kernel of π_f is $ker(\pi_f) = \{\vec{v} \in V \mid \pi_f(\vec{v}) = \vec{0}\} = {V_2}^f$ and the restriction ${\pi_f}^{V_1}$ of π_f to V_1 is the identity 1_{V_1} in V_1. Conversely, let $\pi_f : V \to V$ be a linear operator on V. If the dimension of $im(\pi_f) = V_1$ is d_1 and the dimension of $ker(\pi_f) = {V_2}^f$ is d_2 then $d_1 + d_2 = d$ is the dimension of V. If in addition ${\pi_f}^{V_1} = 1_{V_1}$, that is to say the restriction ${\pi_f}^{V_1}$ of π_f to V_1 is the identity 1_{V_1} in V_1, then $V_1 \cap {V_2}^f = \{\vec{0}\}$. It then follows that $V = V_1 \oplus {V_2}^f$. It again is clear that $\pi_f \circ \pi_f = \pi_f$, which thus makes up another equivalent definition of a projection operator π_f. A bijective correspondence is thus established between the projection operators π_f of V onto V_1 and the complements ${V_2}^f = ker(\pi_f)$ of V_1 in V.

Let $\rho : G \to GL(V, \mathbb{C})$ be a linear representation of a finite group G on a finite-dimensional vector space V over the field $\mathbb{C}$ of the complex numbers. Let V_1 be a proper subspace of the representation space V, which is invariant under the group G. Let ${V_2}^f$ be an arbitrary complement of V_1 in V, not necessarily invariant under the group G. Let π_f be the projection operator of V onto V_1 bijectively associated to ${V_2}^f$. Let π be the "average" of π_f over G, which is defined as:

$$\pi = \frac{1}{n_G} \sum_{g\in G} \rho(g) \circ \pi_f \circ \rho(g^{-1}) \tag{2.19}$$

where n_G is the order of the group G. π is a linear operator on V, since it is a function sum of functionally composed linear operators on V. π "commute" with G:

$$\begin{aligned} \pi \circ \rho(h) &= \frac{1}{n_G} \sum_{g\in G} \rho(g) \circ \pi_f \circ \rho(g^{-1}h) = \frac{1}{n_G} \sum_{g\in G} \rho(hg) \circ \pi_f \circ \rho((hg)^{-1}h) \\ &= \frac{1}{n_G} \sum_{g\in G} \rho(h) \circ \rho(g) \circ \pi_f \circ \rho(g^{-1}h^{-1}h) \;=\; \rho(h) \circ \pi \quad \forall h \in G \end{aligned} \tag{2.20}$$

by using the dummy transformation $g \mapsto hg$ in the second equality and the identity $(hg)^{-1} = g^{-1}h^{-1}$ in the third equality. It follows that $V_2 = ker(\pi)$ is a subspace stable under G: $\forall h \in G$, $\forall \vec{v}_2 \in V_2$, $\pi(\rho(h)(\vec{v}_2)) = \rho(h)(\pi(\vec{v}_2))$, but $\pi(\vec{v}_2) = \vec{0}$ by definition of V_2, so that $\rho(h)(\pi(\vec{v}_2)) = \vec{0}$, whence $\pi(\rho(h)(\vec{v}_2)) = \vec{0}$, that is to say $\rho(h)(\vec{v}_2) \in V_2$. Next,

$$(\pi \circ \pi_f)^{V_1} = 1_{V_1} \quad \text{and} \quad ker(\pi \circ \pi_f) = ker(\pi_f) = {V_2}^f \quad \text{so that} \quad \pi \circ \pi_f = \pi_f \tag{2.21}$$

Indeed, ${\pi_f}^{V_1} = 1_{V_1}$ since π_f is a projection operator and $\vec{v}_1 \in V_1 \Rightarrow \rho(g^{-1})(\vec{v}_1) \in V_1$ by the invariance of V_1 under G, so that $\forall \vec{v}_1 \in V_1$, $\rho(g) \circ \pi_f \circ \rho(g^{-1}) \circ \pi_f(\vec{v}_1) = \rho(g) \circ \pi_f \circ \rho(g^{-1})(\vec{v}_1) = \rho(g) \circ \rho(g^{-1})(\vec{u}_1) = \vec{v}_1$, ergo $\forall \vec{v}_1 \in V_1$, $(\pi \circ \pi_f)(\vec{v}_1) = \vec{v}_1$, that is to say $(\pi \circ \pi_f)^{V_1} = 1_{V_1}$. π_f and π are linear operators and ${V_2}^f = ker(\pi_f)$ so that $\forall \vec{v}_2 \in {V_2}^f$ $(\pi \circ \pi_f)(\vec{v}_2) = \pi(\vec{0}) = \vec{0}$, which means $ker(\pi_f) \subseteq ker(\pi \circ \pi_f)$. If $(\pi \circ \pi_f)(\vec{u}_1) = \vec{0}$ and $\vec{u}_1 \in V_1$ then $\vec{u}_1 = \vec{0}$ since $(\pi \circ \pi_f)^{V_1} = 1_{V_1}$, which implies that

$ker(\pi \circ \pi_f) \subseteq ker(\pi_f)$ since $V = V_1 \oplus V_2{}^f$. It finally is inferred that π is a projection operator:

$$\pi \circ \pi = \pi \circ \frac{1}{n_G} \sum_{g \in G} \rho(g) \circ \pi_f \circ \rho(g^{-1}) = \frac{1}{n_G} \sum_{g \in G} \rho(g) \circ \pi \circ \pi_f \circ \rho(g^{-1}) = \pi \tag{2.22}$$

by using the equation 2.20 in the second equality and the equation 2.21 in the third equality. Accordingly, the G-invariant subspace $V_2 = ker(\pi)$ is a complement of the initially assumed G-invariant subspace V_1: $V = V_1 \oplus V_2$.

A fundamental theorem is thus proven, the so-called **Maschke's Theorem**, which states that ***whatever the linear representation $\rho : G \to GL(V, \mathbb{C})$ of a finite group G on a finite-dimensional vector space V over the field $\mathbb{C}$, to every invariant subspace $V_1 \subseteq V$ is associated an invariant complement $V_2 \subseteq V$.*** With the same proof arguments it is extended, for any finite group G, to any finite-dimensional vector space V over any scalar field $\mathbb{K}$ of any characteristic $char(\mathbb{K})$ that does not divide the order n_G of the group G, this merely by generalizing the average procedure in equation (2.19) to $\mathbb{K}$-summation and division by $n_G 1_{\mathbb{K}}$, where $1_{\mathbb{K}}$ is the multiplicative unit of $\mathbb{K}$. It is clear that if $n_G \equiv 0 \pmod{char(\mathbb{K})}$ then this G-averaging cannot be defined since $n_G 1_{\mathbb{K}} = 0_{\mathbb{K}}$, where $0_{\mathbb{K}}$ is the additive unit of $\mathbb{K}$.

2.5 Inner products

Another proof of Maschke's Theorem can be forged using inner products, inspiring generalizations to compact continuous groups G. An **inner product** on a vector space V over the field $\mathbb{C}$ designates a two-arguments application $\langle \circ \mid \bullet \rangle : V \times V \to \mathbb{C}$, which is i- linear in the second argument ($\bullet$): $\langle \vec{u} \mid a\ \vec{v} + b\ \vec{w} \rangle = a\ \ \langle \vec{u} \mid \vec{v} \rangle + b\ \ \langle \vec{u} \mid \vec{w} \rangle\ \forall (a, b) \in \mathbb{C}^2\ \forall (\vec{u}, \vec{v}) \in V^2$, ii- conjugate symmetric: $\langle \vec{u} \mid \vec{v} \rangle = \langle \vec{v} \mid \vec{u} \rangle^\star\ \ \forall (\vec{u}, \vec{v}) \in V^2$ and iii- positive definite: $\langle \vec{v} \mid \vec{v} \rangle > 0\ \forall \vec{v} \in V - \{\vec{0}\}$. It immediately follows from the two first properties (i and ii) that the inner product is antilinear in the first argument ($\circ$): $\langle a\ \vec{v} + b\ \vec{w} \mid \vec{u} \rangle = a^\star \langle \vec{v} \mid \vec{u} \rangle + b^\star \langle \vec{w} \mid \vec{u} \rangle$, $\forall (a, b) \in \mathbb{C}^2$, $\forall (\vec{u}, \vec{v}) \in V^2$. An inner product in other words is a positive definite conjugate symmetric sesquilinear form.

A **sesquilinear form** is the generic name for any application $\varphi : V \times V \to \mathbb{C}$ which is antilinear in the first argument and linear in the second argument. φ uniquely defines an antilinear application $\varsigma : V \to V^\#$, $\vec{u} \mapsto \vec{u}^\# \equiv \varphi(\vec{u}, \bullet)$. Conversely, an antilinear application from a vector space V to its dual $V^\#$ uniquely determines a sesquilinear form. φ is **non degenerate** iff ς is injective, which means $ker(\varsigma) = \{\vec{0}\}$ or $\varphi(\vec{u}, \vec{v}) = 0\ \forall \vec{v} \in V \Leftrightarrow \vec{u} = \vec{0}$. The sesquilinear form χ defined as $\chi(\vec{u}, \vec{v}) = \varphi(\vec{v}, \vec{u})^\star\ \forall (\vec{u}, \vec{v}) \in V^2$ is the **conjugate symmetric** to φ. If $\chi = \varphi$ ($\chi = -\varphi$) then φ is called an **hermitian form** (anti-hermitian form).

A vector $\vec{u}$ is orthogonal to a vector $\vec{v}$ with respect to a sesquilinear form φ iff $\varphi(\vec{u}, \vec{v}) = 0$. Let W be a subspace of V. The set $W^\perp = \{\vec{u} \in V \mid \forall \vec{v} \in W,\ \varphi(\vec{u}, \vec{v}) = 0\}$ makes up a subspace of V, called the **orthogonal** to W with respect to φ in V. If $\mathbf{W \cap W^\perp = \{\vec{0}\}}$ then the restriction φ^W of φ to W is non degenerate, which means that the restriction ς^W of ς to W is injective. If, in addition, **W is finite-dimensional** then $W^\#$ is of the same dimension as W and ς^W becomes a bijection. ς sends every $\vec{v} \in V$ to a unique linear form $\vec{v}^\# \in V^\#$, since it is an application. The restriction $\vec{w}^\#$ of $\vec{v}^\#$ to W obviously is also unique. To the linear form $\vec{w}^\#$ finally corresponds a unique $\vec{w} \in W$, because ς^W is a bijection. In other words, to every $\vec{v} \in V$ is associated a unique $\vec{w} \in W$ such that $\varphi(\vec{v}, \vec{u}) = \varphi(\vec{w}, \vec{u})\ \forall \vec{u} \in W$, that is to say $\vec{v} - \vec{w} \in W^\perp$. It follows that $V = W \oplus W^\perp$. $W^\perp$ then is called the **orthocomplement** of W in V.

A sesquilinear form φ is positive definite iff $\varphi(\vec{v}, \vec{v}) > 0\ \forall \vec{v} \in V - \{\vec{0}\}$, in which case $\varphi(\vec{v}, \vec{v}) = 0 \Leftrightarrow \vec{v} = \vec{0}$ whence $W \cap W^\perp = \{\vec{0}\}$ whatever the finite-dimensional subspace W of V. Thus, ***to every finite-dimensional subspace W of a vector space V over the field $\mathbb{C}$ endowed with an inner product is associated an orthocomplement $W^\perp$ in V.***

Let α be a **linear operator** on the vector space V. The **transpose** of α is the linear operator ${}^t\alpha$ on the dual space $V^\#$ defined from the pointwise relation ${}^t\alpha(\vec{u}^\#)(\vec{v}) = \vec{u}^\#(\alpha\vec{v})$. ${}^t\alpha(\vec{u}^\#)$ is called the pullback of $\vec{u}^\#$

along α. If α is invertible then ${}^t\alpha = (\alpha^{-1})^\#$. Let φ be a **non degenerate sesquilinear form**. A linear operator $\alpha^\dagger$ may be defined in V from the pointwise relation $\varphi(\vec{u}, \alpha\vec{v}) = \varphi(\alpha^\dagger\vec{u}, \vec{v})$. It is called the **adjoint of** α with respect to φ. If the application $\varsigma : \mathrm{V} \to \mathrm{V}^\#,\ \vec{u} \mapsto \vec{u}^\# \equiv \varphi(\vec{u}, \bullet)$ is bijective, which is the case only if the vector space V is finite-dimensional, then the adjoint of α always exists, given as $\alpha^\dagger = \varsigma^{-1} \circ {}^t\alpha \circ \varsigma$.[6] A sesquilinear form φ by definition is **invariant** with respect to a linear operator α iff $\varphi(\alpha\vec{u}, \alpha\vec{v}) = \varphi(\vec{u}, \vec{v})\ \forall(\vec{u}, \vec{v}) \in \mathrm{V}^2$. Obviously this is the case iff α is invertible and $\alpha^\dagger\alpha = 1_\mathrm{V}$, namely $\alpha^\dagger = \alpha^{-1}$. α then is said **unitary**. The unitary operators are normal operators. A linear operator η is **normal** iff it commutes with its adjoint: $\eta \circ \eta^\dagger = \eta^\dagger \circ \eta$. It is diagonalizable and its eigen-spaces are pairwise orthogonal (spectral theorem for the normal operators). Another subfamily of normal operators are the self-adoint operators: $\kappa^\dagger = \kappa$.[7]

If V is finite-dimensional and $\{\hat{e}_i\}_{i=1,\ldots,d}$ is the selected basis in V then $\forall(\vec{u}, \vec{v}) \in \mathrm{V}^2$, $\exists!\ (u_1, \ldots, u_d) \in \mathbb{C}^d$ s.t. $\vec{u} = \Sigma_i u_i \hat{e}_i$ and $\exists!\ (v_1, \ldots, v_d) \in \mathbb{C}^d$ s.t. $\vec{v} = \Sigma_j v_j \hat{e}_j$, so that $\varphi(\vec{u}, \vec{v}) = \Sigma_{i,j} u_i^\star\ \varphi(\hat{e}_i, \hat{e}_j)\ v_j = \mathfrak{U}^\dagger \Phi \mathfrak{V}$, where $\mathfrak{U}^\dagger (\equiv {}^t\mathfrak{U}^\star)$ is the complex conjugate row vector $(u_1^\star, \ldots, u_d^\star)$ and $\mathfrak{V}$ the column vector $(v_1, \ldots, v_d)$. The sesquilinear matrix Φ with the entries $\Phi_{ij} = \varphi(\hat{e}_i, \hat{e}_j)$ uniquely determines φ once the basis is given. φ is non degenerate iff $\mathrm{Det}(\Phi) \neq 0$. A **basis** $\{\hat{e}_i\}_{i=1,\ldots,d}$ **is orthonormal** with respect to φ iff $\Phi = \mathbb{I}_d$ ($d \times d$ unit matrix). Let α be a **linear operator** and denote $\mathcal{A}$ the matrix representative of α and $\mathcal{A}^\dagger$ the matrix representative of $\alpha^\dagger$ in the $\{\hat{e}_i\}_{i=1,\ldots,d}$ basis. The pointwise relation $\varphi(\vec{u}, \alpha\vec{v}) = \varphi(\alpha^\dagger\vec{u}, \vec{v})$ is transcribed into ${}^t\mathfrak{U}^\star\ \Phi \mathcal{A}\, \mathfrak{V} = {}^t(\mathcal{A}^\dagger\mathfrak{U})^\star\ \Phi\, \mathfrak{V} = {}^t\mathfrak{U}^\star\ {}^t(\mathcal{A}^\dagger)^\star\ \Phi\, \mathfrak{V}$. It follows that $\Phi\ \mathcal{A} = {}^t(\mathcal{A}^\dagger)^\star\ \Phi$ therefore ${}^t\mathcal{A}^\star\ {}^t\Phi^\star = {}^t\Phi^\star\ \mathcal{A}^\dagger$ or else $\mathcal{A}^\dagger = ({}^t\Phi^\star)^{-1}\ {}^t\mathcal{A}^\star\ ({}^t\Phi^\star)$, since by hypothesis φ is non-degenerate. If in addition the chosen basis is orthonormal with respect to φ then $\mathcal{A}^\dagger = {}^t\mathcal{A}^\star$.

It is emphasized that inner products can be defined solely on vector spaces over the field $\mathbb{R}$ of the real numbers, which is an ordered field, or the field $\mathbb{C}$ of the complex numbers, which is not ordered but makes up an ordered extension of the field $\mathbb{R}$. The basic reason is that otherwise it becomes meaningless to require that a sesquilinear form be positive definite. This clearly excludes all the fields with non zero characteristic, which cannot have an ordered subfield.

2.6 Unitarity and unitarisability

A linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ of a finite group G on a vector space V over the field $\mathbb{C}$ by definition is a **unitary representation** if the representation space V is endowed with an inner product $\langle \circ \mid \bullet \rangle : \mathrm{V} \times \mathrm{V} \to \mathbb{C}$ which is invariant under G:

$$\langle \rho(g)(\vec{u}) \mid \rho(g)(\vec{v}) \rangle = \langle \vec{u} \mid \vec{v} \rangle \quad \forall(\vec{u}, \vec{v}) \in \mathrm{V}^2,\ \forall g \in \mathrm{G} \tag{2.23}$$

which means that the linear operators $\rho(g)$ are unitary for every g in G. Another way telling the same thing is that the linear representation ρ commutes with the inner product $\langle \circ \mid \bullet \rangle$.

[6] It is customary **in physics** to use the so-called **bra-ket notation**. The space V then is endowed with an inner product $\langle \circ \mid \bullet \rangle$ (pre-Hilbert space). V is complete for the associated norm (Hilbert space), namely every Cauchy sequence in V converges within V. A vector is denoted by a **ket** $|\Psi\rangle$ and a linear form by a **bra** $\langle \Xi|$. The application of a **linear operator** $\mathcal{O}$ on a ket is described as $\mathcal{O}|\Psi\rangle$. Its **dual** is applied on a bra $\langle \Xi|$ as $\langle \Xi|\mathcal{O}^\#$ to mean $(\langle \Xi|\mathcal{O}^\#)|\Psi\rangle = \langle \Xi|(\mathcal{O}|\Psi\rangle)$. Its **adjoint** is applied on a ket $|\Xi\rangle$ as $\mathcal{O}^\dagger|\Xi\rangle$ so that $\langle \Psi|(\mathcal{O}^\dagger|\Xi\rangle) = (\langle \Xi|\mathcal{O}^\#)|\Psi\rangle^\star$. To any ket $|\Psi\rangle$ one may associate a bra $\langle \Psi|$ (Riez Theorem). The converse is true solely in finite dimension. If V is infinite-dimensional then V can be put in bijection only with the subspace of continuous linear forms in the dual $\mathrm{V}^\#$. The "discontinuous" bra have no ket counterpart.

[7] A bijective correspondence exists between the self-adjoint operators $\mathcal{H}$ on a Hilbert space V and the families of unitary operators $\mathcal{U}(\tau)_{\tau\in\mathbb{R}}$ on V with the group property $\mathcal{U}(\tau + \sigma) = \mathcal{U}(\tau) \circ \mathcal{U}(\sigma)$ and the continuity property $\mathcal{U}(\tau \to \sigma) \to \mathcal{U}(\sigma)$, to be precise $\mathcal{U}(\tau) = \exp(i\tau\mathcal{H})$ (Stone's theorem). When the Hilbert space is separable it suffices to assume weak measurability instead of continuity (von Newman). This bijection is useful in establishing the uniqueness of the irreducible unitary representation of the algebra of canonical commutation relations on finitely many generators (Stone-von Newman theorem). This is no more the case with infinitely many generators, concretely in quantum field theory where in general there is no unitary equivalence between canonical commutation relation representation of the free field and that of the interacting fields (Haag theorem).

If the representation space V is **finite-dimensional** with dimension d then a unitary matrix representation $\Upsilon : \mathrm{G} \to \mathrm{GL}(\mathrm{d}, \mathbb{C})$ of the group G is obtained by selecting in V an **orthonormal basis** $\{\hat{e}_i\}_{i=1,\ldots,d}$ with respect to the inner product $\langle \circ \mid \bullet \rangle$. Υ associates each element g of the group G to a **unitary matrix** representative $\Upsilon(g)$:

$$\Upsilon(g)^\dagger\ \Upsilon(g) = ({}^t\Upsilon(g)^\star)\ \Upsilon(g) = \mathbb{I}_d \quad \forall g \in \mathrm{G} \tag{2.24}$$

where $\mathbb{I}_d$ is the $d \times d$ unit matrix, or else $\Upsilon(g)^\dagger = \Upsilon(g^{-1})$ for every element g in the group G.

Let W be a finite-dimensional proper subspace of the representation space V and let $\mathrm{W}^\perp$ be the orhocomplement of W in V. One has $\vec{v} \in \mathrm{W}^\perp \Leftrightarrow \langle \vec{v} \mid \vec{w} \rangle = 0\ \forall \vec{w} \in \mathrm{W}$ and $\mathrm{V} = \mathrm{W} \oplus \mathrm{W}^\perp$, by definition of $\mathrm{W}^\perp$. Assume that **W is invariant under G**. It follows, by the equation (2.23) that $\vec{v} \in \mathrm{W}^\perp \Leftrightarrow \langle \rho(g)(\vec{v}) \mid \rho(g)(\vec{w}) \rangle = 0\ \forall \vec{w} \in \mathrm{W}$ for every $g \in \mathrm{G}$ or else, choosing $\vec{u} = \rho(g^{-1})(\vec{w})$ and making use of the G-invariance of W, $\vec{v} \in \mathrm{W}^\perp \Leftrightarrow \langle \rho(g)(\vec{v}) \mid \vec{u} \rangle = 0\ \forall \vec{u} \in \mathrm{W}$, which merely means that $\mathbf{W}^\perp$ **is invariant under G**. A sub-representation of a unitary representation is obviously unitary for the restricted inner product. Accordingly, ***every unitary representation of a finite group G on a vector space V over the field*** $\mathbb{C}$ ***that contains a finite-dimensional subspace W invariant under G can be decomposed into two unitary subrepresentations*** as

$$\rho = \rho_{\mathrm{W}} \oplus \rho_{\mathrm{W}^\perp} \tag{2.25}$$

where ρ_{W} stands for the restriction of ρ to W and $\rho_{\mathrm{W}^\perp}$ for the restriction of ρ to the orthocomplement $\mathrm{W}^\perp$ of W in V. The two subrepresentations might in turn be decomposed into subrepresentation and so on. The process must end after a finite number of iterations if V is finite-dimensional, since by hypothesis the invariant subspace is a proper subspace so that at each step the dimension of the subrepresentation spaces to consider is decreased. It nevertheless is emphasized that no conditions is imposed on **the dimension of the representation V**, which thus **might be infinite**. So, at least **as far as G is finite**, the dichotomy processes might go on indefinitely and lead to infinite direct sums or even direct integrals. As a matter of fact, the construction of a meaningful direct integral often can fail, all the more as the group G is unspecified, and leads to extremely delicate and difficult problems of functional analysis.

A linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ is **unitarisable** by definition if an inner product invariant under G can be defined in the representation space V. Assume that V possesses a basis $\{\hat{e}_i\}$. Whatever the vector $\vec{u} = \sum_i x_i(\vec{u})\hat{e}_i$ in V the set of complex numbers $\{x_i(\vec{u})\}$ is uniquely defined. So is the product $\sum_i x_i^\star(\vec{u}) x_i(\vec{v}) = \langle \vec{u} \mid \vec{v} \rangle$, which thus determines an application $\mathrm{V} \times \mathrm{V} \to \mathbb{C}$, conjugate symmetric, linear in the second argument and positive definite ($\sum_i \mid x_i(\vec{u}) \mid^2 > 0$ unless $x_i(\vec{u}) = 0\ \forall i$). In other words, an inner product $\langle \circ \mid \bullet \rangle$ in V is defined by declaring that the basis $\{\hat{e}_i\}$ is orthonormal.[8] If the group **G is finite** then the application

$$\langle \circ \mid \bullet \rangle_G : \mathrm{V} \times \mathrm{V} \to \mathbb{C},\ (\vec{u}, \vec{v}) \mapsto \langle \vec{u} \mid \vec{v} \rangle_G = \sum_{g \in \mathrm{G}} \langle \rho(g)(\vec{u}) \mid \rho(g)(\vec{v}) \rangle \tag{2.26}$$

can always be defined. It is straightforwardly shown that i- $\langle \circ \mid \bullet \rangle_G$ is linear in the second argument because $\langle \circ \mid \bullet \rangle$ is linear in the second argument and $\rho(g)$ is a linear operator on V for every $g \in \mathrm{G}$, ii- $\langle \circ \mid \bullet \rangle_G$ inherits from $\langle \circ \mid \bullet \rangle$ the conjugate symmetry property, and iii- $\langle \circ \mid \bullet \rangle_G$ is positive definite

[8] Every regular representation ρ_{G} of a finite group G for instance is obviously unitarisable. It is made unitary merely by declaring that the G-indexed basis vectors $\hat{e}_h$ are orthonormal in the representation space.

because the sum of strictly positive numbers is strictly positive. It further is found out that

$$\begin{aligned}\langle \rho(g)(\vec{u}) \mid \rho(g)(\vec{v})\rangle_G &= \sum_{h\in \mathrm{G}} \langle \rho(h)\{\rho(g)(\vec{u})\} \mid \rho(h)\{\rho(g)(\vec{v})\}\rangle \\ &= \sum_{h\in \mathrm{G}} \langle \rho(hg)(\vec{u}) \mid \rho(hg)(\vec{v})\rangle = \sum_{k\in \mathrm{G}} \langle \rho(k)(\vec{u}) \mid \rho(k)(\vec{v})\rangle \\ &= \langle \vec{u} \mid \vec{v}\rangle_G \qquad \forall g \in \mathrm{G},\ \forall(\vec{u},\vec{v}) \in \mathrm{V}^2 \end{aligned} \tag{2.27}$$

In other words, $\langle \circ \mid \bullet\rangle_G$ is an inner product which is invariant under G. The linear representation ρ becomes a unitary representation by endowing the representation space V with the inner product $\langle \circ \mid \bullet\rangle_G$. Note that every change of inner products is equivalent to a basis change.[9]

A fundamental theorem is thus proven, which states that ***every linear representation of a finite group G on a vector space V over the field $\mathbb{C}$ is unitarisable and therefore isomorphic to a unitary representation.*** It thus can always be decomposed into subrepresentations whenever there exists a finite-dimensional proper subspace invariant under G in the representation space. The group average displayed in the equation (2.26) is the so-called **Weyl's Trick**. It already was employed in a disguised manner for a projection operator in the equation (2.19). It can be extended to linear representations of topological groups,[10] provided the summation over the group elements can be generalized to an appropriate integration.[11]

One finally may wonder whether the unitarity concept is worth extending to invariance with respect to hermitian forms not necessarily positive definite, to deal with linear representations on vector spaces

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[9] A basis $\{\hat{f}_i\}$ orthonormal with respect to $\langle \circ \mid \bullet\rangle_G$ can even always be built, using for instance the Gram-Schmidt procedure:

$$\hat{f}_i = \frac{\vec{s}_i}{\sqrt{\langle \vec{s}_i \mid \vec{s}_i\rangle_G}} \quad \text{with} \quad \hat{s}_1 = \hat{e}_1 \quad \text{and} \quad \vec{s}_n = \hat{e}_i - \sum_{j=1}^{n} \frac{\langle \hat{e}_n \mid \hat{s}_j\rangle_G}{\langle \hat{s}_j \mid \hat{s}_j\rangle_G}\hat{s}_j \quad (n > 1) \tag{2.28}$$

Of course, the change from the basis $\{\hat{e}_i\}$ to the basis $\{\hat{f}_i\}$ describes nothing but a similarity transformation.

[10] A topological group by definition is a set G endowed with a group structure and a topological structure such that the group operation $\mathcal{G}_{op} : (g,h) \mapsto gh^{-1}$ is a continuous function, to be precise the inverse image of any open set of G by this function is an open set of the topological product space G × G. A topological space is separated iff for any pair of distinct points there exists disjoint neighborhoods (Hausdorff). It is quasi-compact iff a finite cover can be extracted from every open cover (Borel-Lebesgue). It is compact iff it is separated and quasi-compact. It is locally compact iff every point possesses a compact neighborhood. It is simply connected iff every loop is homotopic to the null loop. A loop is a continuous function $\gamma : [0,1] \to G$ such that $\gamma(0) = \gamma(1)$. A loop at a point g is null iff $im(\gamma) = \{g\}$. A loop γ is homotopic to a loop ζ iff there exists a continuous function $\Omega : [0,1] \times [0,1] \to G$ such that $\Omega(0,\sigma) = \Omega(1,\sigma)\ \forall\sigma$ and $\Omega(\tau,0) = \gamma(\tau),\ \Omega(\tau,1) = \zeta(\tau)\ \forall\tau$. A topological group is m-connected iff at every point it shows m homotopy classes of loops. Its representations then might be m-valued, but for each multiply-connected group there exists a simply connected group, the universal cover, that is homomorphic to it. A few examples: $SU(n)$ is compact simply connected. $SO(n)$ is compact 2-connected and its universal cover is $Spin(n)$. $Spin(3)$ is isomorphic to $SU(2)$. $O(p,q)$ $(0 < p \leq q)$ is non-compact 4-connected. . . .

A field is topological iff its additive and multiplicative groups are topological. A vector space on a topological field endowed with a topological structure such that the vector addition and the scalar multiplication are continuous is topological. A continuous representation of a topological group G on a topological vector space V over the field $\mathbb{C}$ is a linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V},\mathbb{C})$ such that the function $r : \mathrm{G} \times \mathrm{V} \to \mathrm{V},\ (g,\vec{v}) \mapsto r(g,\vec{v}) = \rho(g)(\vec{v})$ is continuous on the two variables $g \in \mathrm{G}$ and $\vec{v} \in \mathrm{V}$.

[11] If G is a locally compact topological group then there always exist a measure dg and only one carried by G and enjoying the properties i- $\int_{\mathrm{G}} \mathfrak{F}(g)dg = \int_{\mathrm{G}} \mathfrak{F}(gh)dg$ for every h in G and every continuous function $\mathfrak{F}$ on G (invariance of dg under right translation) and ii- $\int_{\mathrm{G}} dg = 1$ (mass normalization). If G is compact then dg is also invariant under left translation: $\int_{\mathrm{G}} \mathfrak{F}(g)dg = \int_{\mathrm{G}} \mathfrak{F}(hg)dg$, in which case dg is called the bi-invariant or Haar measure of G. If the group G is finite of order n_{G}, the measure dg is obtained by assigning to each g in G a mass equal to $1/n_{\mathrm{G}}$. If G is the group SO(2) of the planar rotations and if every $g \in \mathrm{SO}(2)$ is represented in the form $g \equiv \exp(i\theta)$ (θ taken modulo 2π) the invariant measure is $d\theta/2\pi$. As a matter of fact, the concrete construction of the Haar measure generally is far from being obvious, except possibly for groups of geometric nature $(O(n,\mathbb{K}), SO(n,\mathbb{K}), U(n,\mathbb{K}), \ldots)$. An efficient method can be worked out for a Lie group G of dimension n represented by unitary matrices $\mathcal{U} = \exp(i\mathcal{H})$ of order N. The hermitian matrix $\mathcal{H}$ belongs to the associated Lie algebra $\mathcal{G}$ and can be parametrized as $\mathcal{H}(x) = \sum_p x_p\mathcal{X}_p$ with $x_q = \mathrm{Tr}(\mathcal{H}\mathcal{X}_q)$, by means of the generators $\mathcal{X}_p$ chosen such that $\left[\mathcal{X}_p,\ \mathcal{X}_q\right] = iC_{pqr}\mathcal{X}_r$ and $\mathrm{Tr}(\mathcal{X}_p\mathcal{X}_q) = \delta_{pq}$. As from the invariant metric $\mathrm{Tr}(d\mathcal{U}^\dagger d\mathcal{U}) = -\mathrm{Tr}\left[\mathcal{U}^{-1}d\mathcal{U}\mathcal{U}^{-1}d\mathcal{U}\right] = \Psi_{pq}(x)dx_pdx_q$

over fields with non zero characteristic. A more generalized approach might even be considered, since sesquilinear forms might be defined on any module over a ring for an unspecified antiautomorphism (in place of the conjugate complex involution). The drawback is that the crucial result according to which every proper subspace possesses an orthocomplement then would be lost. Isotropic subspaces, the vectors of which are all orthogonal to at least one of their own non null vectors, might exist, that thus might not necessarily have a complement.

2.7 Irreducibility and reduction

A linear representation of a group is said **irreducible** if its representation space contains no proper invariant subspace under the action of the group and **reducible** otherwise. A reducible representation is not necessarily **decomposable** into subrepresentations, since this requires that to the identified invariant subspace is associated an invariant complement. A linear representation then might be **reducible but indecomposable**. A linear representation is said **completely reducible** if it is decomposable down to irreducible components.

Let $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{K})$ and $\xi : \mathrm{G} \to \mathrm{GL}(\mathrm{W}, \mathbb{K})$ be two linear representations intertwined with the isomorphism $\sigma : \mathrm{V} \to \mathrm{W}$. Assume that there exists a G-invariant subspace V_1 in V and denote W_1 its image by σ in W. W_1 obviously is a subspace of W, which is G-invariant: $\vec{w}_1 \in \mathrm{W}_1 \Rightarrow \sigma^{-1}(\vec{w}_1) \in \mathrm{V}_1 \Rightarrow (\rho \circ \sigma^{-1})(\vec{w}_1) \in \mathrm{V}_1 \Rightarrow (\sigma \circ \rho \circ \sigma^{-1})(\vec{w}_1) = \xi(g)(\vec{w}_1) \in \mathrm{W}_1$. It follows that every linear representation isomorphic to a reducible linear representation is itself reducible. If V_2 is a G-invariant complement to V_1 in V then its image W_2 by σ is a complement of W_1 in W. Indeed, the restriction of σ to V_i $(i = 1,\ 2)$ defines two isomorphisms $\sigma_i : \mathrm{V}_i \to \mathrm{W}_i$ $(i = 1,\ 2)$ so that $\vec{v} \in \mathrm{V}_1 \cap \mathrm{V}_2 \Leftrightarrow \sigma(\vec{v}) \in \mathrm{W}_1 \cap \mathrm{W}_2$ and the dimensions of V_i and W_i $(i = 1,\ 2)$ are the same. W_2 of course is also G-invariant. This means that every linear representation isomorphic to a decomposable linear representation is itself decomposable. Assume now that there is no G-invariant subspace V_1 in V then obviously there can be no invariant subspace in W, otherwise its image by σ^{-1} would be a G-invariant subspace in V in contradiction with the hypothesis. Accordingly, every linear representation isomorphic to an irreducible linear representation is itself irreducible. It similarly is shown that every linear representation isomorphic to a reducible but indecomposable linear representation is itself reducible but indecomposable and every

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and the identity $d(e^{\mathcal{K}}) = \int_0^1 dz e^{(1-z)\mathcal{K}} d\mathcal{K} e^{z\mathcal{K}}$ it is inferred that

$$\int_{\mathrm{G}} \mathfrak{F}(\mathcal{U})\, d\mathcal{U} = \int_X \sqrt{\mathrm{Det}(\Psi(x))}\, \mathfrak{F}(x)\, dx \quad \text{with} \quad \Psi_{pq}(x) = \int_{-1}^{1} (1 - |z|) \exp(z \sum_r x_r C_{rqp})\, dz$$

$\Psi(x)$ is diagonalized by the same unitary matrix as the $n \times n$ real antisymmetric matrix $\mathcal{M}(x) = -\mathcal{M}^{\dagger}(x)$ with the entries $\mathcal{M}_{pq}(x) = \sum_r x_r C_{rqp}$. It follows that if $\pm i\lambda_j$ $(\lambda_j \in \mathbb{R}^+)$ denotes the eigenvalues of the matrix $\mathcal{M}(x)$ then

$$\mathrm{Det}(\Psi(x)) = \prod_{\lambda_i} \frac{\sin^2(\lambda_i/2)}{(\lambda_i/2)^2}$$

The eigenvalue problem $\mathcal{M}(x)v = i\lambda v$ is equivalent to solving the equation $[\mathcal{V}, \mathcal{H}(x)] = \lambda\mathcal{V}$ with $\mathcal{V} = \sum_p v_p \mathcal{X}_p$. It is observed that $\mathcal{S}^{\dagger}(\lambda\mathcal{V})\mathcal{S} = \mathcal{S}^{\dagger}(\mathcal{V}\mathcal{H}(x) - \mathcal{H}(x)\mathcal{V})\mathcal{S} = \mathcal{S}^{\dagger}(\mathcal{V}\mathcal{S}\mathcal{S}^{\dagger}\mathcal{H}(x) - \mathcal{H}(x)\mathcal{S}\mathcal{S}^{\dagger}\mathcal{V})\mathcal{S}$. Thus, if $\mathcal{S}$ is the matrix that diagonalizes $\mathcal{H}(x)$ then $(\mathcal{S}^{\dagger}\mathcal{V}\mathcal{S})_{ij}(\nu_i - \nu_j - \lambda) = 0$: the eigenvalues λ_k of $\mathcal{M}(x)$ needed to evaluate the Haar measure are differences of eigenvalues ν_i of $\mathcal{H}(x)$.

Assume that $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ is a linear representation of a compact group G and assume that the representation space is endowed with an inner product $\langle \circ \mid \bullet \rangle$. The quantity $\langle \vec{u} \mid \vec{v} \rangle_G = \int_{\mathrm{G}} \langle \rho(g)(\vec{u}) \mid \rho(g)(\vec{v}) \rangle \; dg$ (Weyl's Trick) is well defined since G is compact and $g \mapsto \langle \rho(g)(\vec{u}) \mid \rho(g)(\vec{v}) \rangle$ is continuous. It is clearly Hermitian and it is G-invariant since the Haar measure is right invariant. It finally is positive definite: $\langle \vec{v} \mid \vec{v} \rangle_G = \int_{\mathrm{G}} \langle \rho(g)(\vec{v}) \mid \rho(g)(\vec{v}) \rangle \; dg > 0$, $\forall \vec{v} \neq \vec{0}$ since $\langle \rho(g)(\vec{v}) \mid \rho(g)(\vec{v}) \rangle > 0$. We thus have demonstrated that ***every linear representation of a compact group is unitary.*** Using similar arguments as with the unitary representation of finite groups it then is shown that ***every finite-dimensional linear representation of a compact group is completely reducible.*** As a matter of fact, as far as only the finite-dimensional representations on the vector spaces over the field $\mathbb{C}$ are considered, almost all the theorems that are proved for finite groups are safely extended to compact groups, be it that at some places a sum must be replaced by an integral.

linear representation isomorphic to a completely reducible linear representation is itself completely reducible. It is the usage also to call **irreducible** (resp. reducible and decomposable, reducible but indecomposable, completely reducible) the **matrix representation** obtained from an irreducible (resp. reducible and decomposable, reducible but indecomposable, completely reducible) linear representation by selecting a basis in the representation space.

Complete Reducibility Theorems may be formulated for certain families of linear representations. Among the most important for the physics of the finite groups of symmetry is the one which states that ***every linear representation of a finite group on a finite-dimensional vector space over the field of complex numbers is completely reducible.*** As to prove it one proceeds by induction on the dimension d of the representation space V. Assume that the statement holds for all the representations of dimension smaller than d, and let ρ be a linear representation of dimension d. If V is irreducible, then there is nothing to prove. Otherwise, there exists a proper subspace V_1, therefore of dimension $d_1 < d$, invariant under G. According to the **Maschke's Theorem**, V_1 has in V a complement V_2, therefore of dimension $d_2 < d$, which is also invariant under G. Accordingly, $\rho = \rho_1 \oplus \rho_2$, where ρ_i $(i = 1, 2)$ is the restriction of ρ to V_i $(i = 1, 2)$. Now, by the induction hypothesis the subrepresentation ρ_i $(i = 1, 2)$ is completely reducible, since $d_i < d$ $(i = 1, 2)$. So the same is true of ρ, which ends the proof. Note that although the mathematical induction might suggest that the theorem might be true for infinite countable dimension, the corresponding extension would make up an abuse at this step for the Maschke's Theorem is demonstrated only for finite-dimensional V.

The theorem is straightforwardly extended to the linear representation of the finite groups on the finite-dimensional vector spaces over the fields whose characteristic does not divide the order of the group, from the corresponding extension of the Maschke's Theorem. Using the Weyl's Trick the theorem also is extended to the linear representation of the compact groups on the finite-dimensional vector spaces over the field $\mathbb{C}$. Note, meanwhile, that the finite groups are compact, for the discrete topology. It happens that finally the infinite-dimensional case does not cause excessively more troubles for compact groups. It indeed is shown that ***every continuous representation of a compact group on a Hilbert space V, be it infinite-dimensional, is isomorphic to the Hilbert sum of finite-dimensional unitary representations*** and ***the set of G-finite vectors is dense in V.*** A Hilbert sum of unitary representations $\rho_\alpha : \mathrm{G} \to \mathrm{V}_\alpha$ is the unitary representation $\hat{\bigoplus}_\alpha \rho_\alpha : \mathrm{G} \to \hat{\bigoplus}_\alpha \mathrm{V}_\alpha = \{(\vec{v}_\alpha) \mid \vec{v}_\alpha \in \mathrm{V}_\alpha \wedge \sum_\alpha \|\vec{v}_\alpha\|_\alpha^2 < \infty\}$ on the Hilbert sum of the representation spaces V_α, that coincides with ρ_α on each α sector. $\hat{\bigoplus}_\alpha \mathrm{V}_\alpha$ is the Hilbert space with inner product $((\vec{u}_\alpha), (\vec{v}_\alpha)) = \sum_\alpha \langle \vec{u}_\alpha, \vec{v}_\alpha \rangle_\alpha$ and contains $\oplus_\alpha \mathrm{V}_\alpha$ as a dense subspace with $\mathrm{V}_\alpha \perp \mathrm{V}_{\forall\beta\neq\alpha}$. A set of G-finite vectors is the set of all vectors $\vec{v}_{fin}$ in V such that the dimension of the vector space spanned by $\{\rho(g)(\vec{v}_{fin}),\ g \in \mathrm{G}\}$ is finite. It follows in particular that the ***irreducible unitary representations of the compact groups are all finite dimensional.*** A proof is provided first by showing that there always exists a finite-dimensional G-invariant (closed) subspace in V, for instance the eigenspace of any non zero eigenvalue of a G-averaged compact operator on V, and next, using the Zorn's Lemma, by establishing that the set $\hat{\bigoplus}_\alpha \mathrm{V}_\alpha$, partially ordered by inclusion, necessarily shows a maximal element. As a result $\hat{\bigoplus}_\alpha \mathrm{V}_\alpha$ cannot be different from V, otherwise there would exist $\mathrm{V}_\gamma \in (\hat{\bigoplus}_\alpha \mathrm{V}_\alpha)^\perp$ in violation of the maximality. Note that the "Zorn's Lemma" is equivalent to the axiom of choice (see footnote 5). Non compact groups do show infinite-dimensional representations which are more delicate to handle or else linear representations that cannot be isomorphic to unitary representations or reducible representations that are indecomposable.[12]

[12] Although to some extent either exotic or pathological for what might concern physical systems the counterexamples to the complete reducibility of the linear representations are not that uncommon, even with finite groups, and it always is instructive to have scrutinized at least one. Consider for instance the matrix representation

$$\Gamma : \mathcal{C}_p = \langle s \mid s^p = e \rangle \to \mathrm{GL}(2, \mathbb{Z}/q\mathbb{Z}),\ s^k \mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

of the cyclic group $\mathcal{C}_p$ of order p and generator s on the linear group of the 2×2 invertible matrices with entries in the field $\mathbb{Z}/q\mathbb{Z}$ of characteristic $char(\mathbb{Z}/q\mathbb{Z}) = q$. At first it is observed that if q does not divide p then Γ cannot be a group homomorphism and

Now, let $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ be a completely reducible linear representation of a finite group G. Choose an initial G-invariant subspace, find its complement and perform a first decomposition into two sub-representations, then proceed similarly on each of these and so on until getting only irreducible sub-representations. Grouping isomorphic irreducible summands, one most generally would write $\rho = \xi_1 \oplus \xi_2 \oplus \ldots \oplus \xi_s = \bigoplus_k \xi_k$, where ξ_k is isomorphic to the direct sum of n_k copies of an irreducible linear representation $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$, these by construction being non-isomorphic for different k's. A symbolic manner transcribing all this is

$$\mathrm{V} \cong \bigoplus_k \mathrm{V}_k^{\oplus n_k} \quad \text{and} \quad \rho \sim n_1\rho_1 \oplus n_2\rho_2 \oplus \cdots \oplus n_s\rho_s \equiv \bigoplus_k n_k\rho_k \tag{2.29}$$

where $\mathrm{V}_k^{\oplus n_k}$ is isomorphic to the subspace X_k of V spanned by the different G-invariant subspaces of V associated with each copy of ρ_k and n_k defines the **multiplicity** of the irreducible component ρ_k contained in ρ. It is customary to call $\rho = \bigoplus_k \xi_k$ the **canonical decomposition** of ρ, or else the decomposition of ρ into **isotypical components** ξ_k. An irreducible matrix representation $\Gamma^k : \mathrm{G} \to \mathrm{GL}(\mathrm{d_k}, \mathbb{C})$ is associated with the irreducible linear representation $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$ as soon as a basis is selected in the representation space V_k. With every isomorphism of V that transforms a given copy of V_k in V to another copy of V_k in V is associated two distinct bases in one-to-one correspondence and two isomorphic irreducible matrix representations. A basis of $\mathrm{X}_k \cong \mathrm{V}_k^{\oplus n_k}$ thus may be built from different isomorphisms in V sending an initial copy of V_k in V to the different copies of V_k in V. With respect to this basis the linear representation ξ_k is associated to a matrix representation $\Lambda^k : \mathrm{G} \to \mathrm{GL}(\mathrm{n_k d_k}, \mathbb{C})$ isomorphic to the direct sum of n_k copies of the irreducible matrix representations $\Gamma^k : \mathrm{G} \to \mathrm{GL}(\mathrm{d_k}, \mathbb{C})$. A basis in V is obtained from the union of the bases built on each subspace X_k, since V is the direct sum of the $\mathrm{X}_k \cong \mathrm{V}_k^{\oplus n_k}$. The matrix representation $\Gamma : \mathrm{G} \to \mathrm{GL}(\mathrm{d}, \mathbb{C})$ associated with the linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ with respect to this basis in V is given as the direct sum $\Gamma = \Lambda^1 \oplus \Lambda^2 \oplus \ldots \oplus \Lambda^s = \bigoplus_k \Lambda^k$. It again is standard to write

$$\Gamma \sim n_1\Gamma^1 \oplus n_2\Gamma^2 \oplus \cdots n_s\Gamma^s = \bigoplus_k n_k\Gamma^k \tag{2.30}$$

and customary to call $\Gamma = \bigoplus_k \Lambda^k$ the **canonical decomposition** of Γ, or else the decomposition of Γ into **isotypical components** Λ^k. A similar procedure may be replicated to get canonical decompositions of linear representations of compact groups, possibly by using Hilbert sums of representations. Note that at this stage it is not sure whether the canonical decomposition is unique, so deserves its name, and whether the n_k are unambiguously defined.

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therefore cannot be a matrix representation associated with a linear representation. Next $ker(\rho) = \{e\}$, that is to say ρ is injective, iff $q = p$. Now assuming that either q divides p or equals p, the one-dimensional space spanned by the $(1, 0)$ vector is invariant under $\mathcal{C}_p$, but it has no invariant complement: **the representation is reducible but indecomposable**. In a different context, if l is a prime then the set $\mathbb{Z}_l = inv.lim.\mathbb{Z}/l^n\mathbb{Z}$ of l-adic integers makes up a compact topological group, which has the continuous reducible but indecomposable representation

$$\rho : \mathbb{Z}_l \to \mathrm{GL}(2, \mathbb{Q}_l),\ x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

on a 2-dimensional vector space over the field $\mathbb{Q}_l$ of l-adic numbers. This example tells that "compact group" and "continuous representation" are not enough conditions. The basis field must be $\mathbb{C}$. Substituting the additive group $\mathbb{R}$ for $\mathbb{Z}_l$ and the automorphism group $\mathrm{GL}(2, \mathbb{C})$ for $\mathrm{GL}(2, \mathbb{Q}_l)$ a third example of continuous representation is obtained, which again is reducible but indecomposable. It also is not unitarizable. In this case the failure of complete reducibility is to ascribe to the fact that $\mathbb{R}$ is not compact. It is only locally compact, because it is not bounded. The compact subsets of $\mathbb{R}^n$ ($\mathbb{C}^n$) are the closed and bounded subsets of $\mathbb{R}^n$ ($\mathbb{C}^n$).

2.8 Schur's lemmas

It is clear that there exists a number of ways to decompose reducible linear representations down to irreducible components, so that to proceed further it is necessary to get deeper insights into their isomorphisms. As a matter of fact, the irreducible linear (or matrix) representations are special in their intertwining. This is formulated in the **Schur's Lemmas**:

Let $\rho_1 : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_1, \mathbb{C})$ and $\rho_2 : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_2, \mathbb{C})$ be two irreducible representations of a finite group G and let $\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1, \mathrm{V}_2) = \{\sigma : \mathrm{V}_1 \to \mathrm{V}_2 \mid \rho_2(g) \circ \sigma = \sigma \circ \rho_1(g)\, \forall g \in \mathrm{G}\}$ be the vector space of intertwining operators from V_1 to V_2. Then, denoting $d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)}$ the dimension of $\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1, \mathrm{V}_2)$,

$$\textbf{Schur 1} - \rho_1 \sim \rho_2 \Longleftrightarrow d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)} = 1$$

$$\textbf{Schur2} - \rho_1 \nsim \rho_2 \Longleftrightarrow d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)} = 0$$

A proof is provided by observing that $ker(\sigma) = \{\vec{v} \in \mathrm{V}_1 \mid \sigma(\vec{v}) = \vec{0}_2\}$ is a G-invariant subspace of V_1 and $im(\sigma) = \{\sigma(\vec{v}) \mid \vec{v} \in \mathrm{V}_1\}$ is a G-invariant subspace of V_2: $\vec{v} \in ker(\sigma) \Rightarrow \sigma(\rho_1(\vec{v})) = \rho_2(\sigma(\vec{v})) = \rho_2(\vec{0}_2) = \vec{0}_2 \Rightarrow \rho_1(\vec{v}) \in ker(\sigma)$ and $\vec{v} \in im(\sigma) \Rightarrow \exists \vec{u} \in \mathrm{V}_1 : \sigma(\vec{u}) = \vec{v} \Rightarrow \exists \vec{w} = \rho_1(\vec{u}) \in \mathrm{V}_1 : \sigma(\vec{w}) = \sigma(\rho_1(\vec{u})) = \rho_2(\sigma(\vec{u})) = \rho_2(\vec{v}) \Rightarrow \rho_2(\vec{v}) \in im(\sigma)$. The irreducibility of ρ_1 and ρ_2 leaves $ker(\sigma) = \{\vec{0}_1\}$ or V_1 and $im(\sigma) = \mathrm{V}_2$ or $\{\vec{0}_2\}$ as the only options. σ is non zero iff $ker(\sigma) = \{\vec{0}_1\}$, which means that σ is injective, and $im(\sigma) = \mathrm{V}_2$, which means that σ is surjective, that is to say iff σ is an isomorphism. As a consequence, $\rho_1 \sim \rho_2 \Leftrightarrow \mathrm{V}_1 \cong \mathrm{V}_2 \Leftrightarrow \mathrm{Hom}_\mathrm{G}(\mathrm{V}_1, \mathrm{V}_2) \neq \{\mathbf{0}\}$, which partially proves Schur 1, and $\rho_1 \nsim \rho_2 \Leftrightarrow \mathrm{V}_1 \ncong \mathrm{V}_2 \Leftrightarrow \mathrm{Hom}_\mathrm{G}(\mathrm{V}_1, \mathrm{V}_2) = \{\mathbf{0}\}$, whence $d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)} = 0$, which ends the proof of Schur 2.

$\mathrm{V}_1 \cong \mathrm{V}_2 \Leftrightarrow \mathrm{Hom}_\mathrm{G}(\mathrm{V}_1, \mathrm{V}_2) \cong \mathrm{End}_\mathrm{G}(\mathrm{V}_1) \cong \mathrm{End}_\mathrm{G}(\mathrm{V}_2)$, where $\mathrm{End}_\mathrm{G}(\mathrm{V}_i)$ $(i = 1, 2)$ is the vector space of the endomorphisms σ_i $(i = 1, 2)$ of V_i $(i = 1, 2)$ that commute with $\mathrm{G} : \rho_i(g) \circ \sigma_i = \sigma_i \circ \rho_i(g)$ $(i = 1, 2)\, \forall g \in \mathrm{G}$. Unlike $\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1, \mathrm{V}_2 \ncong \mathrm{V}_1)$, which is only a vector space, $\mathrm{End}_\mathrm{G}(\mathrm{V}_i)$ $(i = 1, 2)$, endowed with the canonical composition law $\circ$ for the functions, shows the structure of a division algebra, with unit ϵ_i $(i = 1, 2)$ and composition inverse for each of its non zero elements. Now, select a non zero σ_1 in $\mathrm{End}_\mathrm{G}(\mathrm{V}_1)$ and pick up another arbitrary $\tau \in \mathrm{End}_\mathrm{G}(\mathrm{V}_1)$. Obviously $\tau \circ \sigma_1^{-1} \in \mathrm{End}_\mathrm{G}(\mathrm{V}_1)$. It is implicitly assumed that the representation space V_1 is finite-dimensional. Accordingly, as the field $\mathbb{C}$ is **algebraically closed**, there always exists for $\tau \circ \sigma_1^{-1}$ an eigenvalue $\lambda \in \mathbb{C}$: $\ker(\tau \circ \sigma_1^{-1} - \lambda\epsilon_1) \neq \{\vec{0}\}$. On the other hand, $[(\tau \circ \sigma_1^{-1} - \lambda\epsilon_1) \circ \rho_1(g)](\vec{v}) = [\rho_2(g) \circ (\tau \circ \sigma_1^{-1} - \lambda\epsilon_1)](\vec{v}) = \vec{0}$, $\forall g \in \mathrm{G}$ and $\forall \vec{v} \in \ker(\tau \circ \sigma_1^{-1} - \lambda\epsilon_1)$: $\ker(\tau \circ \sigma_1^{-1} - \lambda\epsilon_1)$ is a G-invariant subspace of V_1. The irreducibility of ρ_1 then implies that $\tau \circ \sigma_1^{-1} - \lambda\epsilon_1 = \mathbf{0}$, that is to say $\tau = \lambda\sigma_1$ or else $\mathrm{End}_\mathrm{G}(\mathrm{V}_1) \equiv \mathbb{C}\sigma_1$. In other words $\rho_1 \sim \rho_2$ iff every intertwining operator from V_1 to V_2 is isomorphic to an endomorphism of V_1 proportional to σ_1, whence $d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)} = 1$, which ends the proof of Schur 1.

Schur's Lemma are straightforwardly generalized to finite-dimensional irreducible representations of compact groups, using the same proof arguments. With infinite-dimensional representations discrete eigenvalues might not necessarily exist and one has to resort to the spectral theorem for normal bounded operators, which states that for any σ in $\mathrm{End}_\mathrm{G}(\mathrm{V})$ there exists a projection valued measure μ such that $\sigma = \int_{\mathrm{spec}(\sigma)} \lambda d\mu$ and that the only bounded endomorphisms of V commuting with σ are the ones commuting with the self-adjoint projection $\mu(\mathcal{B})$ for each Borel subset $\mathcal{B}$ of the spectrum $\mathrm{spec}(\sigma)$. Whatever the case, Schur 1 obviously implies that

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C}) \text{ is irreducible } \Longleftrightarrow \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{V}) \equiv \mathrm{End}_\mathrm{G}(\mathrm{V}) \cong \mathbb{C}\, 1_\mathrm{V}$$

Schur's Lemma may be extended to scalar fields $\mathbb{K}$ other than the field $\mathbb{C}$ of complex numbers under the weaker formulation:

$$\rho_1 \sim \rho_2 \Longleftrightarrow d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)} = d_{\mathrm{End}_\mathrm{G}(\mathrm{V}_i,\mathbb{K})\ (i=1,2)} \neq 0$$

$$\rho_1 \nsim \rho_2 \Longleftrightarrow d_{\mathrm{Hom}_\mathrm{G}(\mathrm{V}_1,\mathrm{V}_2)} = 0$$

which is inferred solely from the G-invariance of the subspaces $ker(\sigma)$ and $im(\sigma)$ for any σ in $\mathrm{Hom_G}(\mathrm{V}_1, \mathrm{V}_2)$ and the irreducibility of ρ_1 and ρ_2. It also is clear that any non zero σ in $\mathrm{Hom_G}(\mathrm{V}_1, \mathrm{V}_2)$ is an isomorphism and therefore $\mathrm{Hom_G}(\mathrm{V}_1, \mathrm{V}_2 \cong \mathrm{V}_1) \cong \mathrm{End_G}(\mathrm{V}_i, \mathbb{K})$ $(i = 1, 2)$, endowed with the canonical composition law $\circ$ for the functions, shows the structure of a division algebra over the field $\mathbb{K}$. This leads to three possibilities: i- if $\mathbb{K}$ is **algebraically closed** then $d_{\mathrm{End_G}(\mathrm{V}_i,\mathbb{K})\ (i=1,2)} = 1$ and $\mathrm{End_G}(\mathrm{V}_i, \mathbb{K})$ $(i = 1, 2) \cong \mathbb{K}\ 1_\mathrm{V} \cong \mathbb{C}\ 1_\mathrm{V}$, ii- if $\mathbb{K}$ is **real closed**, that is to say if $\mathbb{K}$ is not algebraically closed but its closure is a finite extension, then by virtue of the (1,2,4,8)-Theorem on the real division algebras and since it implicitly is clear that $\mathrm{End_G}(\mathrm{V}_i, \mathbb{K})$ $(i = 1, 2)$ is associative but not necessarily commutative, $d_{\mathrm{End_G}(\mathrm{V}_i,\mathbb{K})\ (i=1,2)}$ may take the values $1, 2, 4$ and the division algebra $\mathrm{End_G}(\mathrm{V}_i, \mathbb{K})$ $(i = 1, 2)$ may be isomorphic to either $\mathbb{R}\ 1_\mathrm{V}, \mathbb{C}\ 1_\mathrm{V}$ or $\mathbb{Q}\ 1_\mathrm{V}$, where $\mathbb{Q}$ stands for the field of quaternions.[13] iii- if $\mathbb{K}$ is **neither algebraically closed nor real closed** then $d_{\mathrm{End_G}(\mathrm{V}_i,\mathbb{K})\ (i=1,2)}$ is the square of an integer.

The transcription of the Schurs Lemmas into the language of complex matrix representations of finite groups is easily inferred as:

Schur 1 - If $\Gamma : \mathrm{G} \to \mathrm{GL(d,\mathbb{C})}$ is an irreducible complex matrix representation of dimension d of a finite group G then every $d \times d$ matrix A commuting with Γ is a multiple of the $d \times d$ identity matrix $\mathbf{1}_d$:

$$\{\Gamma \text{ irreducible}\} \ \wedge \ \{\Gamma(g)\ A = A\ \Gamma(g)\ \forall g \in \mathrm{G}\} \ \Rightarrow \ \exists \lambda \in \mathbb{C} : \ A = \lambda\ \mathbf{1}_d$$

Schur 2 - No intertwining may exist between two irreducible complex matrix representation of a finite group G except if these are associated with isomorphic representation spaces:

$$\{\Gamma^{1,2} \text{ irreducible}\} \ \wedge \ \{\Gamma^2(g)\ A = A\ \Gamma^1(g)\ \forall g \in \mathrm{G}\} \ \Rightarrow \ \{A = \ \mathbf{0} \text{ or } \Gamma^1 \sim \Gamma^2\}$$

Schur's Lemmas have a number of impacting outcomes. Schur 1 for instance implies that ***every irreducible complex representation*** $\rho : \mathrm{G} \to \mathrm{GL(V, \mathbb{C})}$ ***of an abelian group G is*** 1***-dimensional***: $\forall g \in \mathrm{G}$, since G is abelian, $\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g)\ \forall h \in \mathrm{G}$, whence, since ρ is irreducible, $\exists \lambda_g \in \mathbb{C} : \rho(g) = \lambda_g 1_\mathrm{V}$, by Schur 1. It follows that $\forall \vec{v} \in \mathrm{V}\ \rho(g)(\vec{v}) = \lambda_g \vec{v} \Rightarrow \rho(g)(\vec{v}) \in Span(\vec{v})$, that is to say every 1-dimensional subspace $Span(\vec{v}) = \{a\ \ \vec{v}\ \mid\ a \in \mathbb{C}\}$ of V necessarily is G-invariant. The irreducibility of ρ then implies that the representation space V itself is 1-dimensional. This easily is generalized to compact groups using similar arguments,[14] but fails with scalar fields $\mathbb{K}$ that are not algebraically closed. A simple illustration is provided by the real representations $\rho : \mathcal{C}_3 = \langle s |\, s^3 = e \rangle \to \mathrm{GL(V, \mathbb{R})}$ of the cyclic group $\mathcal{C}_3$. If ρ is irreducible then it either is isomorphic to the 1-dimensional trivial representation or to the 2-dimensional representation that associates the generator s of $\mathcal{C}_3$ to the 2-dimensional geometric rotation by an angle $2\pi/3$ in a plane. The matrix representative of this rotation with respect to any selected basis in V has complex eigenvalues. It thus

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[13] The (1,2,4,8)-Theorem can be given different equivalent formulations. It in particular states that, up to isomorphism, the only division algebra over a real closed field are the 1-dimensional real algebra $\mathbb{R}$, the 2-dimensional complex algebra $\mathbb{C}$, the 4-dimensional quaternion algebra $\mathbb{Q}$ and the 8-dimensional octonion algebra $\mathbb{O}$. At each increase of the algebra dimension an essential property is lost: a nonidentical involution must be introduced to get $\mathbb{C}$, commutativity is lost with $\mathbb{Q}$ then associativity is lost with $\mathbb{O}$, but these algebra still are alternative. Algebras of higher dimension are constructed using the dimension-doubling Cayley-Dickson process: $(x_1, x_2)(y_1, y_2) = (x_1 y_1 - y_2 x_2^\star, x_1^\star y_2 + y_1 x_2)$, $(x_1, x_2)^\star = (x_1^\star, -x_2)$. According to this, the next in the list is the 16-dimensional sedenion algebra $\mathbb{S}$, which is no more alternative nor a division algebra, but retains the property of power associativity. The (1,2,4,8)-Theorem encompasses the weaker previous Frobenius', Hurwitz's and Zorn's Theorems on the real division algebras, but unlike these is not proved algebraically. It actually emerges as a corollary to a theorem of topological nature: the existence of an arbitrary division algebra of dimension n over the reals implies parallelizability of the sphere S^{n-1} but according to the Bott-Milnor-Kervaire Theorem spheres are parallelizable only in dimensions $n = 1, 2, 4, 8$ (a manifold is parallelizable iff the tangent space at each point stay isomorphic to its transform induced by any parallel transport along a curve). There exists a variety of other avatars of the (1,2,4,8)-Theorem, in Topology (Hopf bundles over the spheres $\mathrm{S}^n, \ldots$), in Geometry (construction of exceptional Lie algebra, ...), in Number Theory (a sum of n squares of integers times another sum of n squares of integers is a sum of n squares of integers iff $n = 1, 2, 4, 8, \ldots$), ...

[14] A number of way exists to establish that all the irreducible representations of a compact group are 1-dimensional iff G is abelian. One may use for instance the fact that the commutator group $C_\mathrm{G} = \{ghg^{-1}h^{-1} \mid g, h \in \mathrm{G}\} = \{e\}$ iff G is abelian and that this acts trivially on 1-dimensional representations.

cannot be diagonalized with only real entries in the diagonals. As a matter of fact, it can be shown that the irreducible representations $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{K})$ of an abelian group G are 1-dimensional over the endomorphism ring $\mathrm{End}_{\mathrm{G}}(\mathrm{V}, \mathbb{K})$, which makes up an extension field of the field $\mathbb{K}$.

Schur 2 allows demonstrating that ***the canonical decomposition of completely reducible linear representations is unique***. Let $\rho = \bigoplus_k \xi_k$ and $\pi = \bigoplus_k \zeta_k$ be canonical decompositions of two linear representations $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ and $\pi : \mathrm{G} \to \mathrm{GL}(\mathrm{U}, \mathbb{C})$. Any σ in $\mathrm{Hom}_{\mathrm{G}}(\mathrm{U}, \mathrm{V})$ maps the representation space $\mathrm{Z}_k \cong \mathrm{U}_k^{\oplus m_k}$ of ζ_k to the representation space $\mathrm{X}_k \cong \mathrm{V}_k^{\oplus n_k}$ of ξ_k, because every restriction σ_{kq} of σ from a copy of U_k to a copy of V_q intertwines with two irreducible representations so is null as soon as $k \neq q$ by virtue of Schur 2. In the more intuitive language of matrix representations, if $\Gamma = \bigoplus_k \Lambda^k$ and $\Delta = \bigoplus_k \Upsilon^k$ are two canonical decompositions and if Γ and Δ are intertwined with a matrix S then this cannot contain a non null off-diagonal block $\mathrm{S}_{k,q\neq k}$ with which the isotypical components Λ^k of Γ and $\Upsilon^{q\neq k}$ of Δ would be intertwined. It follows, by taking for π an irreducible representation $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$, that every sub-representation of ρ which is isomorphic to an irreducible representation ρ_k is contained in ξ_k, which gives an intrinsic description of ξ_k as isomorphic to the direct sum of *all* the copies of ρ_k contained in ρ. Accordingly, the canonical decomposition does not depend on the manner it might be performed, which proves its uniqueness.

Another consequence of the Schur's Lemmas, of utmost practical relevance for irreducible matrix representations, is the so-called **Orthogonality Theorem**. Whatever the two irreducible representations $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$ and $\rho_q : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_q, \mathbb{C})$ of a finite group G and the linear application τ from V_q to V_k, the average of τ over the group G, which is defined as

$$\sigma = \frac{1}{n_{\mathrm{G}}} \sum_{g \in \mathrm{G}} \rho_k(g) \circ \tau \circ \rho_q(g^{-1}) \tag{2.31}$$

is an intertwining operator: $\rho_k(h) \circ \sigma = \sigma \circ \rho_q(h)\ \forall h \in \mathrm{G}$.[15] In other words, $\sigma \in \mathrm{Hom}_{\mathrm{G}}(\mathrm{V}_q, \mathrm{V}_k)$. It then follows from the Schur's Lemmas that $\rho_k \sim \rho_q \Leftrightarrow \sigma = \lambda 1_{\mathrm{V}_k \cong \mathrm{V}_q}$ and $\rho_k \nsim \rho_q \Leftrightarrow \sigma = \mathbf{0}$. $\lambda = \mathrm{Tr}\left[\tau\right] / \mathrm{Tr}\left[1_{\mathrm{V}_k}\right]$, since

$$\mathrm{Tr}\left[\sigma\right] = \frac{1}{n_{\mathrm{G}}} \sum_{g \in \mathrm{G}} \mathrm{Tr}\left[\rho_k(g) \circ \tau \circ \rho_q(g^{-1})\right] \overset{\rho_k \sim \rho_q}{=} \frac{1}{n_{\mathrm{G}}} \sum_{g \in \mathrm{G}} \mathrm{Tr}\left[\rho_q(g) \circ \tau \circ (\rho_q(g))^{-1}\right] = \mathrm{Tr}\left[\tau\right]$$

and $\mathrm{Tr}\left[1_{\mathrm{V}_k}\right] = d_k$, where d_k is the dimension of ρ_k. Now, selecting a basis in V_k and a basis in V_q, the linear representations ρ_k and ρ_q and the linear operators τ and σ get associated respectively with matrix representations Γ^k and Γ^q and $d_k \times d_q$ (k lines $-$ q columns) matrices T and S. In terms of matrix elements of the corresponding matrices the equation (2.31) writes:

$$S_{jn} = \frac{1}{n_{\mathrm{G}}} \sum_{g \in \mathrm{G}\ lm} \Gamma^k_{jl}(g)\ T_{lm}\ \Gamma^q_{mn}(g^{-1}) \tag{2.32}$$

which comes out as a linear form with respect to the variables T_{lm}. If $\Gamma^k \nsim \Gamma^q$, that is to say if $k \neq q$, then this form vanishes for all systems of values of the T_{lm}. Its coefficients therefore are null, whence $\sum_{g \in \mathrm{G}} \Gamma^k_{jl}(g)\ \Gamma^q_{mn}(g^{-1}) = 0$ for arbitrary j, l, m, n. If $\Gamma^k \sim \Gamma^q$, that is to say if $k = q$, then $S_{jn} = \lambda \delta_{jn}$

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$$\begin{aligned} \rho_k(h) \circ \sigma \circ (\rho_q(h))^{-1} &= \frac{1}{n_{\mathrm{G}}} \sum_{g \in \mathrm{G}} \rho_k(h) \circ \rho_k(g) \circ \tau \circ (\rho_q(g))^{-1} \circ (\rho_q(h))^{-1} \\ &= \frac{1}{n_{\mathrm{G}}} \sum_{g \in \mathrm{G}} \rho_k(hg) \circ \tau \circ (\rho_q(hg))^{-1} = \sigma \quad \forall h \in \mathrm{G}. \end{aligned}$$

with $\lambda = \mathrm{Tr}\,[\tau]\,/d_k = (1/d_k)\sum_{lm}\delta_{lm}T_{lm}$, whence

$$\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\sum_{lm}\Gamma^k_{jl}(g)\;T_{lm}\;\Gamma^q_{mn}(g^{-1}) = \left(\frac{1}{d_k}\sum_{lm}\delta_{lm}\;T_{lm}\right)\delta_{jn} \tag{2.33}$$

which, by equating the coefficients of the T_{lm}, gives $\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\Gamma^k_{jl}(g)\;\Gamma^q_{mn}(g^{-1}) = \frac{1}{d_k}$ if $l = m$ and $j = n$ and $\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\Gamma^k_{jl}(g)\;\Gamma^q_{mn}(g^{-1}) = 0$ otherwise. All the possibilities are summarized under the compact formula:

$$\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\Gamma^k_{jl}(g)\;\Gamma^q_{mn}(g^{-1}) = \frac{1}{d_k}\;\delta_{kq}\;\delta_{jn}\;\delta_{lm} \tag{2.34}$$

where δ_{kq} stands for a generalized Kronecker symbol, defined as $\delta_{kq} = 1$ if $\Gamma^k \sim \Gamma^q$ and $\delta_{kq} = 0$ if $\Gamma^k \nsim \Gamma^q$. δ_{jn} (resp. δ_{lm}) is the standard Kronecker symbol $\delta_{jn} = 1$ (resp. $\delta_{lm} = 1$) iff $j = n$ (resp. $l = m$) and 0 otherwise. If the matrix representations are unitary then $\Gamma^q_{mn}(g^{-1}) = ((\Gamma^q(g))^{-1})_{mn} = ((\Gamma^q(g))^{\dagger})_{mn} = \Gamma^q_{nm}(g)^{\star}$, which leads to the alternative formula:

$$\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\Gamma^k_{jl}(g)\;\Gamma^q_{nm}(g)^{\star} = \frac{1}{d_k}\;\delta_{kq}\;\delta_{jn}\;\delta_{lm} \tag{2.35}$$

The theorem can be proved also by directly using any pair of irreducible matrix representations Γ^k and Γ^q and applying the Schur's Lemmas to the matrix $A = \sum_{g\in\mathrm{G}}\Gamma^k(g)\;\Xi\;\Gamma^q(g^{-1})$, where Ξ is a $d_k \times d_q$ matrix with entries all null except at line l and column m where it is set to $\Xi_{lm} = 1$. The theorem is straightforwardly extended to the finite-dimensional linear representations of compact groups G on the vector spaces over the field $\mathbb{C}$. It suffices in the proof to replace every normalized sum $\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\ldots$ over a finite group G by the corresponding integration $\int_{\mathrm{G}}\ldots dg$ using the Haar measure dg of the compact group G. It also is extended to every ground field $\mathbb{K}$ whose characteristic $char(\mathbb{K})$ that does not divide the order n_{G} of the group G, except only that $\frac{1}{n_{\mathrm{G}}}\sum_{g\in\mathrm{G}}\Gamma^k_{nm}(g)\;\Gamma^q_{mn}(g^{-1})$ can fail to give $\frac{1}{d_k}$ if $\mathbb{K}$ is not algebraically closed. This can be determined from the Galois Theory of the centre of the division algebra $\mathrm{End}_{\mathrm{G}}(\mathrm{V}, \mathbb{K})$.

3. CHARACTER THEORY

What now one needs are effective methods for reducing a linear representation and constructing the irreducible components of its representation space, to allow discerning the invariances of a physical quantity with respect to a symmetry group. It is obvious from the considerations of the previous sections that, quite quickly, this might become cumbersome. Invariants over the isomorphism classes of the linear representations should be of the greatest help, at the condition that these also allow distinguishing between non isomorphic linear representations.

Whatever the finite dimensional linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ of a compact group G the linear operators $\rho(g)$ for every element g in the group G are diagonalizable, since ρ is unitarisable and unitary operators are diagonalisable with pairwise orthogonal eigenspaces (cf. spectral theorem for normal operators). It is recalled that finite groups are compact, for the discrete topology. As a matter of fact, with finite groups it even may be asserted that all the eigenvalues of $\rho(g)$ are roots of unity, since every element $g \in \mathrm{G}$ necessarily is of finite order, that is to say $\exists n_g : g^{n_g} = e$ so that $\rho(g)^{\bigcirc n_g} = 1_{\mathrm{V}}$. Numerical invariants may be deduced from the symmetric functions of these eigenvalues, more precisely from the coefficients $\alpha_n(g)$ of the characteristic polynomial $\mathrm{Det}[\rho(g) - \lambda 1_V] = (-1)^d\lambda^d + \sum_{n=1}^{d}\alpha_n(g)\lambda^{d-n}$, where d is the dimension of the representation space V. Among the most familiar are the coefficient $\alpha_d(g) = \mathrm{Det}[\rho(g)]$ of the constant term and the coefficient

$\alpha_1(g) = (-1)^{d-1}$ Tr$[\rho(g)]$ of the sub-leading term. It is clear that Det$[\sigma \circ \rho(g) \circ \sigma^{-1}]$ = Det$[\rho(g)]$[16] and Tr$[\sigma \circ \rho(g) \circ \sigma^{-1}]$ = Tr$[\rho(g)]$[17] whatever the invertible linear operator σ on V. Thus, Det$[\rho(g)]$ and Tr$[\rho(g)]$ show the required invariance over every isomorphism class of linear representations. Now, it follows from the multiplicativity of the Determinant that Det$[\rho(g) \circ \rho(h)]$ = Det$[\rho(g)]$Det$[\rho(h)]$, which means that the application $g \in \mathrm{G} \mapsto \mathrm{Det}[\rho(g)] \in \mathbb{C}$ makes up a 1-dimensional representation of G. It thus turns out that the Determinant invariant is often unable to distinguish between different classes of isomorphism when, by contrast, the Trace invariant, which is not multiplicative, can. So this is the searched invariant. It actually will be shown below that the complex-valued function on G defined as

$$\chi : \mathrm{G} \to \mathbb{C},\ g \mapsto \chi(g) = \mathrm{Tr}[\rho(g)] \tag{3.1}$$

is a **complete invariant**, in the sense that it uniquely determines the linear representation $\rho : \mathrm{G} \to \mathrm{GL(V}, \mathbb{C})$ up to isomorphism. χ defines the **character** of the linear representation ρ.

3.1 Elementary properties

Let $\rho : \mathrm{G} \to \mathrm{GL(V}, \mathbb{C})$ be a d-dimensional linear representation of a finite (or even continuous compact) group G and let $\Gamma : \mathrm{G} \to \mathrm{GL(d}, \mathbb{C})$ be the matrix representation associated to ρ with respect to the basis vectors $\hat{e}_m$ $(m = 1, \ldots, d)$ selected in the representation space V. It follows from the definition of the trace of a linear operator that

$$\chi(g) = \mathrm{Tr}\left[\rho(g)\right] = \mathrm{Tr}\left[\Gamma(g)\right] = \sum_i \Gamma_{ii}(g) \quad \forall g \in \mathrm{G} \tag{3.2}$$

It is the usage to also call χ the character of the matrix representation Γ. The trace of a product of matrices being invariant by cyclic permutation, we have $\forall g \in \mathrm{G}$ Tr$[\mathrm{S}\ \Gamma(g)\ \mathrm{S}^{-1}]$ = Tr$[\Gamma(g)]$, whatever the invertible matrix S. Of course, this is nothing but the transposition to the matrix representations of the group G of the invariance of the character χ over an isomorphism class. $\boldsymbol{\chi}$ concretely **is independent of any choice of basis vectors** in the representation space V.

- $\boldsymbol{\chi(e) = d}$, where e is the unit element of G. $\chi(e) = \mathrm{Tr}\left[\Gamma(e)\right] = \mathrm{Tr}\left[\mathbb{I}_d\right] = \sum_{i=1}^{d} 1 = d$, where $\mathbb{I}_d$ is the $d \times d$ unit matrix.
- $\boldsymbol{\chi(g^{-1}) = \chi(g)^\star}$ and $\boldsymbol{\mid \chi(g) \mid \leq d\ \forall g}$ in every **finite group G**. $\forall g \in \mathrm{G}\ \exists n_g \in \mathbb{N} : g^{n_g} = e$ (unit element of G), otherwise the successive powers of g would generate an infinite group. It follows that $\Gamma(g^{n_g}) = \Gamma(g)^{n_g} = \mathbf{1}_d$. It then is directly clear that $\Gamma(g)$ is diagonisable. Let $\epsilon_1(g), \ldots, \epsilon_d(g)$ be the g-dependent eigenvalues of $\Gamma(g)$. Obviously, $\epsilon_i(g)^{n_g} = 1$, which means that $\epsilon_i(g)$ is a root of unity, $\exists\ \varphi_i(g) : \epsilon_i(g) = e^{j\varphi_i(g)}$ with $j = \sqrt{-1}$. Now, $\chi(g)^\star = \mathrm{Tr}\left[\Gamma(g)\right]^\star = \sum_i \epsilon_i(g)^\star = \sum_i \epsilon_i(g)^{-1} = \mathrm{Tr}\left[\Gamma(g)^{-1}\right] = \mathrm{Tr}\left[\Gamma(g^{-1})\right] = \chi(g^{-1})$ whereas $\mid \chi(g) \mid = \mid \mathrm{Tr}\left[\Gamma(g)\right] \mid = \mid \sum_i \epsilon_i(g) \mid \leq \sum_i \mid \epsilon_i(g) \mid = \sum_i 1 = d$. Note that by the theorem of Lagrange the order n_g of g divides the order n_G of the group G. So the eigenvalues $\epsilon_1(g), \ldots, \epsilon_d(g)$ of $\Gamma(g)$ are roots of unity of orders dividing the order n_G of the group G. More generally, every linear representation of a compact group and *à fortiori* of a finite group is unitarisable. An inner product thus may be defined in the representation space V so that $\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^\dagger\ \forall g \in \mathrm{G}$. In terms of matrix representations with respect to

[16] A Determinant most generally designates every alternating d-linear form F: End(M, $\mathbb{A}$) $\to \mathbb{A}$ on the module End(M, $\mathbb{A}$) of the endomorphisms on a free module M of dimension d over a commutative ring $\mathbb{A}$. F is unique up to the image F(1_M) of the identity endomorphism 1_M. One standardly put F(σ)/F(1_M) = Det[σ]. It results from the functorial properties of the exterior algebra on the module M that Det is multiplicative: Det$[\tau \circ \sigma]$ = Det$[\tau]$Det$[\sigma]$ $\forall(\tau, \sigma) \in$ End(M, $\mathbb{A})^2$. As an obvious consequence, the image by Det of any composition of endomorphisms σ_i is invariant by any permutation π of these: Det$[\bigcirc_i \sigma_i]$ = Det$[\bigcirc_i \sigma_{\pi(i)}]$.

[17] A Trace most generally designates every linear form F: End(M, $\mathbb{A}$) $\to \mathbb{A}$ on the module End(M, $\mathbb{A}$), of the endomorphisms on a free module M of dimension d over a commutative ring $\mathbb{A}$, enjoying the property F($\tau \circ \sigma$) = F($\sigma \circ \tau$) $\forall(\tau, \sigma) \in$ End(M, $\mathbb{A})^2$. F is unique up to the image F(1_M) of the identity endomorphism 1_M. One standardly put F(σ)/F(1_M) = Tr$[\sigma]/d$. Obviously, by substituting $\kappa \circ \eta$ for σ and so on, the property F($\tau \circ \sigma$) = F($\sigma \circ \tau$) implies that the Trace of any composition of endomorphisms is invariant under cyclic permutation, whence Tr$[\tau \circ \sigma \circ \tau^{-1}]$ = Tr$[\sigma]$ for invertible linear operators on a vector space. Note that Det$[e^\sigma] = e^{\mathrm{Tr}[\sigma]}$.

an orthonormal basis, this transposes to $\Gamma(g^{-1}) = \Gamma(g)^\dagger = {}^t\Gamma(g)^\star\ \forall g \in \mathrm{G}$ (cf. Sections 2.5 and 2.6), whence $\chi(g^{-1}) = \mathrm{Tr}[\Gamma(g^{-1})] = \mathrm{Tr}[{}^t\Gamma(g)^\star] = \chi(g)^\star\ \forall g \in \mathrm{G}$.

- If $\chi^\#$ is the character of the representation $\rho^\#$ dual to the linear representation ρ with character χ then $\boldsymbol{\chi^\#(g) = \chi(g^{-1})\ \forall g \in \mathrm{G}}$. $\rho^\#(g)$ indeed acts on every linear form on V as the composition with $\rho(g^{-1})$: $\forall \vec{v}^\# \in \mathrm{V}^\# = \mathrm{Hom}(\mathrm{V},\mathbb{C}),\ \ \rho^\#(g)(\vec{v}^\#) = \vec{v}^\# \circ \rho(g)^{-1}$.
- The character of $\boldsymbol{\Gamma = \bigoplus_i \Gamma^i}$ is $\boldsymbol{\chi = \sum_i \chi^i}$, where χ^i stands for the character of Γ^i. Evident from the property $\mathrm{Tr}\,[A \oplus B] = \mathrm{Tr}\,[A] + \mathrm{Tr}\,[B]$ for any pair of matrices A and B.

3.2 Orthogonality theorem

Getting back to the equation 2.34 and setting $j = l$ and $n = m$ then summing over all j and all n and finally using the identity $\sum_{jn}(\delta_{jn})^2 = \sum_{jn}(\delta_{jn}) = d_k$, one ends up at

$$\frac{1}{n_\mathrm{G}} \sum_{g\in\mathrm{G}} (\chi^q(g))^\star\ \chi^k(g) = \langle \chi^q \mid \chi^k \rangle = \delta_{kq} \tag{3.3}$$

where χ^k and χ^q are the characters of the irreducible representations $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$ and ρ_q : $\mathrm{G} \to \mathrm{GL}(\mathrm{V}_q, \mathbb{C})$. δ_{kq} is a generalized Kronecker symbol, defined as $\delta_{kq} = 1$ if $\rho^k \sim \rho^q$ and $\delta_{kq} = 0$ if $\rho^k \nsim \rho^q$. The notation $\langle \varphi \mid \phi \rangle$ is used to emphasize that the quantity $\frac{1}{n_\mathrm{G}} \sum_{g\in\mathrm{G}}(\varphi(g))^\star\ \phi(g)$ does define an inner product in the vector space $\mathbb{C}\,[\mathrm{G}]$ of complex-valued functions on G, being obviously linear with respect to φ, conjugate symmetric and positive definite ($\langle \phi \mid \phi \rangle > 0\ \forall \phi \in \mathbb{C}\,[\mathrm{G}] - \{\mathbf{0}\}$).[18] Equation (3.3) makes up the **First Orthogonality Theorem for the Characters** and has far-reaching consequences.

Consider a decomposition $\rho = \rho_1 \oplus \ldots \oplus \rho_s$ of a linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ with character χ into the irreducible representations $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$ with characters χ_k. It results from the additivity property of the characters that $\chi = \chi^1 + \ldots + \chi^s$ and from the linearity of the inner product that $\langle \chi^q \mid \chi \rangle = \langle \chi^q \mid \chi^1 \rangle + \ldots + \langle \chi^q \mid \chi^s \rangle$. According to the First Orthogonality Theorem for the Characters,

$$\rho_q \sim \rho_k \Longleftrightarrow \langle \chi^q \mid \chi^k \rangle = 1$$

$$\rho_q \nsim \rho_k \Longleftrightarrow \langle \chi^q \mid \chi^k \rangle = 0$$

It follows that $\langle \chi^q \mid \chi \rangle$ determines the number of ρ^k isomorphic to ρ^q contained in the decomposition of ρ. As previously transcribed in the equation (2.29), this number is nothing but the **multiplicity** n_q of ρ^q in the expansion of the representation ρ over its irreducible components ρ^k:

$$\rho \sim \bigoplus_k n_k \rho_k \Longrightarrow n_q = \langle \chi^q \mid \chi \rangle \tag{3.4}$$

The multiplicity of the trivial representation in this expansion for instance is $\sum_{g\in\mathrm{G}} \chi(g)$. Obviously $n_q = \langle \chi^q \mid \chi \rangle$ does not depend on the chosen decomposition, which means that ***the decomposition of a finite-dimensional linear representation of a finite group into irreducible representations is unique***. This in turn immediately implies that ***every two completely reducible linear representations with the same character are necessarily isomorphic***, for they contain each given irreducible representation the same number of times. Characters thus are in one-to-one correspondence with isomorphic classes of linear representations, which is the essence of the **Theorem of Complete Invariance of the Characters**.

[18] Of course, this may be extended to compact groups as $\int_\mathrm{G}(\chi^q(g))^\star\ \chi^k(g)dg = \langle \chi^q \mid \chi^k \rangle = \delta_{kq}$ by using the Haar integration and to every ground field $\mathbb{K}$ whose characteristic $char(\mathbb{K})$ that does not divide the order n_G of the group G, with the proviso to keeping in mind that the square norm $\langle \chi^q \mid \chi^q \rangle$ can fail to give 1 if $\mathbb{K}$ is not algebraically closed, that is to say **we always have orthogonality but not necessarily orthonormality**.

Given that every decomposition of a linear representation ρ uniquely writes $\rho \sim \bigoplus_k n_k \rho_k$ every character uniquely writes $\chi = \sum_k n_k \chi^k$. Computing the square norm of χ and taking account of the First Orthogonality Theorem for the Characters one gets

$$\langle \chi \mid \chi \rangle = \left\langle \sum_q n_q \chi^q \mid \sum_k n_k \chi^k \right\rangle = \sum_{qk} n_q n_k \left\langle \chi^q \mid \chi^k \right\rangle = \sum_k n_k^2 \tag{3.5}$$

$\sum_k n_k^2$ is equal to 1 only if one of the n_k's is equal to 1 and the others to 0, that is if ρ is isomorphic to one of the irreducible representation ρ_k, whence ***if χ is the character of a representation then $\langle \chi \mid \chi \rangle$ is the sum of squares of integers and $\langle \chi \mid \chi \rangle = 1$ iff ρ is irreducible***. We obtain thus a very convenient **irreducibility criterion**.

3.3 Dimensional closure

Consider the **regular representation** ρ_{G} of a finite group G (cf. Section 2.1). ρ_{G} by definition transcribes the left action of the group G on the representation space $\mathrm{V_G}$ spanned by basis vectors $\hat{e}_h$ indexed with the group elements $h \in \mathrm{G}$ by permuting these as $\rho_{\mathrm{G}}(g)(\hat{e}_h) = \hat{e}_{gh}$ $\forall g \in \mathrm{G}$ $\forall h \in \mathrm{G}$. It is clear by the group properties that $gh = h \Leftrightarrow g = e$, where e is the unit element of G. It follows that $\rho_{\mathrm{G}}(g)(\hat{e}_h) = \hat{e}_h \Leftrightarrow g = e$. This means that the diagonal elements of the matrix representatives $\Gamma_{\mathrm{G}}(g)$ of the linear operators $\rho_{\mathrm{G}}(g)$ with respect to the basis $\{\hat{e}_h\}_{h\in\mathrm{G}}$ are all null for $g \neq e$ and all equal to 1 for $g = e$. The character χ_{G} of of the regular representation ρ_{G} then is given by the formula:

$$\chi_{\mathrm{G}}(g) = \begin{cases} n_{\mathrm{G}} \text{ if } g = e \\ 0 \quad \text{otherwise} \end{cases} \tag{3.6}$$

where n_{G} is the order of G. One finds that $\langle \chi_{\mathrm{G}} \mid \chi_{\mathrm{G}} \rangle = \frac{1}{n_{\mathrm{G}}} \sum_{g\in\mathrm{G}} (\chi_{\mathrm{G}}(g))^\star\ \chi_{\mathrm{G}}(g) = \frac{1}{n_{\mathrm{G}}} n_{\mathrm{G}}^2 = n_{\mathrm{G}}$. So ρ_G is far from being irreducible. If χ_q stands for the character of an irreducible representation $\rho_q : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_q, \mathbb{C})$ with dimension d_q of the group G then one also computes

$$n_q = \left\langle \chi^q \mid \chi_{\mathrm{G}} \right\rangle = \frac{1}{n_{\mathrm{G}}} \sum_{g\in\mathrm{G}} (\chi^q(g))^\star\ \chi_{\mathrm{G}}(g) = \frac{1}{n_{\mathrm{G}}} (\chi^q(e))^\star\ \chi_{\mathrm{G}}(e) = \frac{1}{n_{\mathrm{G}}}\ d_q\ n_{\mathrm{G}} = d_q$$

It follows that

$$\rho_{\mathrm{G}} = \bigoplus_k d_k \rho_k \tag{3.7}$$

that is to say ***the number of times each irreducible linear representation ρ_k is contained in the regular representation ρ_G is equal to the dimension d_k of that irreducible representation***. The equation (3.7) implies that $\chi_{\mathrm{G}}(g) = \sum_k d_k \chi^k(g)$ for all g in G. Taking $g = e$ leads to the **dimensional closure** identity

$$\sum_k d_k^2 = n_{\mathrm{G}} \tag{3.8}$$

since $\chi_{\mathrm{G}}(e) = n_{\mathrm{G}}$ and $\chi^k(e) = d_k$. This identity is useful in the determination of the irreducible representations of a group G, to check in particular that all of these have been found out. If $g \neq e$ then, since $\chi_{\mathrm{G}}(g \neq e) = 0$,

$$\sum_k d_k \chi^k(g \neq e) = 0 \tag{3.9}$$

Note that the span $\mathrm{V_G}$ of $\{\hat{e}_h\}_{h\in\mathrm{G}}$ is isomorphic to the vector space $\mathbb{C}\,[\mathrm{G}]$ of complex valued functions on the group G. As to build an isomorphism it suffices to match the basis vector $\hat{e}_h$ in G with the function $\varphi_h : \mathrm{G} \to \mathbb{C},\ g \mapsto \delta_{gh}$. Under this isomorphism the elements g in G act on the left on $\mathbb{C}\,[\mathrm{G}]$ by sending the function φ to the function $\rho_{\mathrm{G}}(g)(\varphi)$ such that $\rho_{\mathrm{G}}(g)(\varphi)(h) = \varphi(g^{-1}h)$. As a matter of fact, this is the way to generalize the concept of regular representations to the **compact groups**. The representation space V_G then is isomorphic to the Hilbert space $\mathcal{L}^2(G, \mathbb{C})$ of the square integrable functions on the group G and $\rho_G(g)$ for each $g \in G$ operates on this space by sending every $\varphi \in \mathcal{L}^2(G, \mathbb{C})$ to $\rho_G(g)(\varphi) \in \mathcal{L}^2(G, \mathbb{C})$ defined as $\rho_G(g)(\varphi)(h) = \varphi(g^{-1}h)\ \forall h \in G$. It again is shown that the number of times each irreducible linear representation ρ_k is contained in the regular representation ρ_G is equal to the dimension d_k of that irreducible representation, but now no dimensional closure prevails since the group G is not finite. The regular representation ρ_G then is infinite-dimensional.

3.4 Class functions

Owing to the invariance $\mathrm{Tr}[\tau \circ \sigma \circ \tau^{-1}] = \mathrm{Tr}[\sigma]$ of the Trace of any pair (τ, σ) of invertible linear operators on any vector space, **the character χ of every linear representation $\rho : G \to GL(V, \mathbb{C})$ is conjugation-invariant**:

$$\chi(tgt^{-1}) = \mathrm{Tr}[\rho(t) \circ \rho(g) \circ \rho(t^{-1})] = \mathrm{Tr}[\rho(g)] = \chi(g) \quad \forall g \in G\ \forall h \in G \tag{3.10}$$

It is recalled that two elements g and h of a group G are conjugate iff there exists another element t in the group G such that $h = tgt^{-1}$. Conjugacy is an equivalence relation that partitions the group G into conjugacy classes $\mathfrak{C}_i$. A complex valued function φ on G is called a **class function** iff $\varphi(tgt^{-1}) = \varphi(g)\ \forall g \in G\ \forall t \in G$, that is to say iff it is constant over each conjugacy class $\mathfrak{C}_i$. It is clear from the equation (3.10) that **every character χ of a linear representation $\rho : G \to GL(V, \mathbb{C})$ of a finite group G is a class function**.

The set of the class functions on a group G, endowed with addition and scalar multiplication makes up a subspace $\mathbb{C}[\mathfrak{C}_G]$ of the vector space $\mathbb{C}[G]$ of the complex valued functions on G. Whatever the linear representation $\rho : G \to GL(V, \mathbb{C})$ of a finite group G and whatever the complex valued function $\varphi \in \mathbb{C}[G]$, we always may define a linear operator on V as:

$$\rho_\varphi = \sum_{g \in G} \varphi(g)\rho(g) \tag{3.11}$$

φ is a class function iff ρ_φ commutes with the group G through any linear representation ρ:[19]

$$\varphi \in \mathbb{C}[\mathfrak{C}_G] \Longleftrightarrow \rho(h) \circ \rho_\varphi = \rho_\varphi \circ \rho(h) \quad \forall h \in G. \tag{3.12}$$

It follows that if φ is a class function and ρ is isomorphic to an irreducible representation $\rho_k : G \to GL(V_k, \mathbb{C})$ of the group G with character χ_k then, by Schur 1, $\exists \lambda \in \mathbb{C} : \rho_\varphi = \lambda 1_{V_k}$ (cf. Section 2.8). λ can be determined by computing $\mathrm{Tr}\left[\rho_\varphi\right]$.[20] As a partial conclusion, we write

$$\varphi \in \mathbb{C}[\mathfrak{C}_G] \quad \text{and} \quad \rho_\varphi \sim \sum_{g \in G} \varphi(g)\rho_k(g) \quad \Longrightarrow \quad \rho_\varphi = \frac{n_G}{d_k} \left\langle \varphi^\star \mid \chi^k \right\rangle 1_{V_k} \tag{3.13}$$

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$$\varphi \in \mathbb{C}[\mathfrak{C}_G] \Rightarrow \rho(h) \circ \rho_\varphi \circ \rho(h^{-1}) = \sum_{g \in G} \varphi(g)\rho(hgh^{-1}) = \sum_{u (= hgh^{-1}) \in G} \varphi(h^{-1}uh)\rho(u) = \rho_\varphi$$

and

$$\rho(h) \circ \rho_\varphi \circ \rho(h^{-1}) = \rho_\varphi \Rightarrow \sum_{u \in G} \varphi(h^{-1}uh)\rho(u) = \sum_{u \in G} \varphi(u)\rho(u) \Rightarrow \varphi(h^{-1}uh) = \varphi(u)$$

Note that the last deduction is obvious if we take for ρ the regular representation ρ_G.

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$$\mathrm{Tr}\left[\rho_\varphi\right] = \mathrm{Tr}\left[\sum_{g \in G} \varphi(g)\rho_k(g)\right] = \sum_{g \in G} \varphi(g)\mathrm{Tr}\left[\rho_k(g)\right] = \sum_{g \in G} \varphi(g)\chi_k(g) = n_G \left\langle \varphi^\star \mid \chi^k \right\rangle \text{ and } \mathrm{Tr}\left[\lambda 1_{V_k}\right] = d_k.$$

where n_G is the order of G and d_k the dimension of V_k. Now, assume that the class function φ is orthogonal to the character χ^k of every irreducible representation ρ_k then, by virtue of the equation (3.13), $\rho_\varphi = \sum_{g\in G} \varphi(g)\rho(g)$ is zero so long as ρ is irreducible and by the decomposition into irreducible representations we conclude that ρ_φ is always zero. Applying this to the regular representation ρ_G and computing the image under ρ_φ of the basis vector $\hat{e}_e$ indexed with unit element e of G, we obtain

$$\rho_\varphi(\hat{e}_e) = \sum_{g\in G} \varphi(g)\rho_G(g)(\hat{e}_e) = \sum_{g\in G} \varphi(g)(\hat{e}_g) \tag{3.14}$$

but $\rho_\varphi(\hat{e}_e) = 0$, since ρ_φ is zero, therefore $\varphi(g)\ \forall g \in G$, whence φ is the null function on G. In short

$$\left\langle \varphi^\star \mid \chi^k \right\rangle = 0\ \forall \chi^k \quad \Longrightarrow \quad \varphi = \mathbf{0} \tag{3.15}$$

It is on the other hand clear from the equation (3.3) that the characters χ^k of the irreducible representations of the group G make up an orthonormal system in the space of the class functions $\mathbb{C}\,[\mathfrak{C}_G]$. In other words ***the characters of the irreducible representations of a finite group G form an orthonormal basis for the space of the complex class functions $\mathbb{C}\,[\mathfrak{C}_G]$***, which is the expression of the **Theorem of Character Completeness over the Class Functions**. Again this is straightforwardly generalized to the compact groups G by using the Haar integration for summation over G and considering the Hilbert space $\mathcal{L}^2(\mathfrak{C}_G, \mathbb{C})$ of the square integrable class functions on G. With the other ground fields $\mathbb{K}$ the application of Schur 1 on the linear operator ρ_φ will involve the division algebra $\mathrm{End}(V_k, \mathbb{K})$.

As an immediate consequence, ***the number of irreducible representations of a finite group G up to isomorphism is equal to the number $n_{\mathfrak{C}}$ of conjugacy classes of G***. Indeed, if $\mathfrak{C}_1, \ldots, \mathfrak{C}_{n_\mathfrak{C}}$ are the distinct conjugacy classes of G then every class function $\varphi \in \mathbb{C}\,[\mathfrak{C}_G]$ is fully determined by its values $\varphi_{\mathfrak{C}_i} \in \mathbb{C}$ on each conjugacy class $\mathfrak{C}_i$. It therefore has $n_\mathfrak{C}$ degrees of freedom. This merely means that the dimension of $\mathbb{C}\,[\mathfrak{C}_G]$ is $n_\mathfrak{C}$, but, by the Character Completeness over the Class Function, this is equal to the number of irreducible representations of G. This is still true of compact groups, but without any interest since there then are infinitely many classes and infinitely many irreducible representations in the group G.

Completeness means that every class function $\varphi \in \mathbb{C}\,[\mathfrak{C}_G]$ on a group G is the linear combination $\varphi = \sum_k \left\langle \chi^k \mid \varphi \right\rangle \chi^k$ of the characters χ^k of the irreducible representations ρ_k of the group G. With the class function φ_g that takes the value 1 for every element of the class $\mathfrak{C}_g = \{h \in G \mid \exists\, t \in G,\ h = tgt^{-1}\}$ and 0 elsewhere, we compute $\left\langle \chi^k \mid \varphi_g \right\rangle = \frac{n_{\mathfrak{C}_g}}{n_G}(\chi^k(g))^\star$ where $n_{\mathfrak{C}_g}$ is the number of elements in the class $\mathfrak{C}_g$ and n_G the order of the group G. It follows, by definition of φ_g, that

$$\frac{n_{\mathfrak{C}_g}}{n_G} \sum_{k=1}^{d_{\mathbb{C}[\mathfrak{C}_G]}} (\chi^k(g))^\star \chi^k(h) = \varphi_g(h) = \begin{cases} 1 \text{ if } h \in \mathfrak{C}_g \\ 0 \text{ if } h \notin \mathfrak{C}_g \end{cases} \tag{3.16}$$

where $d_{\mathbb{C}[\mathfrak{C}_G]}$ is the dimension of $\mathbb{C}\,[\mathfrak{C}_G]$, which, it is recalled, is equal to the number of classes $n_\mathfrak{C}$ in G. Equation 3.16 makes up the **Second Orthogonality Theorem for the Characters**.

3.5 Character tables

Character Orthogonality, Complete Invariance and Completeness over the Class Functions offer the great advantage to allow globally handling all the irreducible linear representations of a finite group G up to isomorphism by means of the so-called Character Table. This is a square matrix with rows labelled by the isomorphism classes of irreducible representations, columns labelled by the conjugacy classes of the group and entries given by the values of the character for each isomorphism class of irreducible representation and for each conjugacy class. Every linear representation of the group can be characterized from this table by determining the multiplicities of its irreducible components from the inner product with the rows of the table and even its decomposition into isotypical components from projection operators on the representation space built over the irreducible characters as discussed in the Section 3.6.

Given a finite group G the first stage to construct its Character Table is to find its conjugacy classes. A series of properties of conjugate elements exist that ease this search. A few of them are:

- ⋆ The unit element e of every group always forms a conjugacy class $\{e\}$ by its own.
- ⋆ In an abelian group every element form a conjugacy class by its own.
- ⋆ The orders of the elements of the same conjugacy class $\mathfrak{C}_i$ are all equal, since obviously $g_i^{n_{g_i}} = e$ and $\exists t \in \mathrm{G} : h_i = t g_i t^{-1} \Rightarrow h_i^{n_{g_i}} = (t g_i t^{-1})(t g_i t^{-1}) \ldots (t g_i t^{-1}) = t g_i^{n_{g_i}} t^{-1} = e$.
- ⋆ If h_i is conjugate to g_i then h_i^{-1} is conjugate to g_i^{-1} so that all the inverses of the elements of a given conjugacy class $\mathfrak{C}_i$ belong to a same conjugacy class $\mathfrak{C}_i^{-1}$. If g_i and g_i^{-1} are conjugate then we a have a single conjugacy class, $\mathfrak{C}_i = \mathfrak{C}_i^{-1}$, which is said ambivalent, otherwise we have two distinct conjugacy classes $\mathfrak{C}_i \neq \mathfrak{C}_i^{-1}$, which are said inverse of each other.
- ⋆ If $n_{\mathfrak{C}_i}$ stands for the number of elements in each conjugacy class $\mathfrak{C}_i$ then, inherently to the partition of the group G into conjugacy classes, we have the class equation $\sum_i n_{\mathfrak{C}_i} = n_{\mathrm{G}}$ where n_{G} is the order of the group G.
- ⋆ The elements of the conjugacy class $\mathfrak{C}_i$ of any given element g_i of the group G are in bijective correspondence with the cosets of the normalizer $\mathrm{N}_{\mathrm{G}}(g_i) = \{t \in \mathrm{G} \mid t g_i t^{-1} = g_i\}$. $\mathrm{N}_{\mathrm{G}}(g_i)$ is a subgroup of G so that $\mathrm{G} = e\,\mathrm{N}_{\mathrm{G}}(g_i) + \ldots + s_j\,\mathrm{N}_{\mathrm{G}}(g_i) + \ldots + s_{[\mathrm{G}:\mathrm{N}_{\mathrm{G}}(g_i)]}\,\mathrm{N}_{\mathrm{G}}(g_i)$, where $[\mathrm{G} : \mathrm{N}_{\mathrm{G}}(g_i)]$ defines the index in G of $\mathrm{N}_{\mathrm{G}}(g_i)$. Conjugating g_i with any element $s_j t$ of the coset $s_j\,\mathrm{N}_{\mathrm{G}}(g_i)$ we get $(s_j t) g_i (s_j t)^{-1} = s_j t g_i (t^{-1} s_j^{-1}) = s_j g_i s_j^{-1}$. On the other hand, if $(s_j t) g_i (s_j t)^{-1} = (s_k r) g_i (s_k r)^{-1}$ then $(s_k r)^{-1} s_j t\ g_i\ (s_j t)^{-1} s_k r = g_i$ so $(s_k r)^{-1} s_j t = h \in \mathrm{N}_{\mathrm{G}}(g_i)$ or else $s_j = s_k(r h t^{-1})$ which means $s_j\,\mathrm{N}_{\mathrm{G}}(g_i) = s_k\,\mathrm{N}_{\mathrm{G}}(g_i)$. It then is inferred that the conjugation of g_i by the elements of distinct cosets leads to distinct conjugates. Thus each conjugate of g_i by an element of the coset $s_j\,\mathrm{N}_{\mathrm{G}}(g_i)$ can be uniquely labelled by this coset as g_i^j. It follows that $n_{\mathfrak{C}_i}$ is the index $[\mathrm{G} : \mathrm{N}_{\mathrm{G}}(g_i)]$ in G of the normalizer of the representative g_i of the conjugacy class $\mathfrak{C}_i$, but by the Lagrange Theorem $[\mathrm{G} : \mathrm{N}_{\mathrm{G}}(g_i)] = n_{\mathrm{G}}/n_{\mathrm{N}_{\mathrm{G}}(g_i)}$. Therefore $n_{\mathfrak{C}_i}$ is a divisor of n_{G}. It is recalled more generally that the normalizer $\mathrm{N}_{\mathrm{G}}(S)$ of a subset S of elements of a group G is defined as $\mathrm{N}_{\mathrm{G}}(S) = \{t \in \mathrm{G} \mid t S t^{-1} = S\}$. A related concept is the centralizer $\mathrm{C}_{\mathrm{G}}(S)$ of the subset S, which is defined as $\mathrm{C}_{\mathrm{G}}(S) = \{t \in \mathrm{G} \mid t S = S t\}$. It goes without saying that, obviously, the normalizer $\mathrm{N}_{\mathrm{G}}(g_i)$ of a single element g_i of the group G is identical to the centralizer $\mathrm{C}_{\mathrm{G}}(g_i) = \{t \in \mathrm{G} \mid t g_i = g_i t\}$ of that element g_i in the group G.
- ⋆ The intersection $\mathrm{Z}(\mathrm{G}) = \cap_{g \in \mathrm{G}} \mathrm{C}_{\mathrm{G}}(g)$ defines the Center of G. Z(G) is an abelian subgroup of G and contains all the elements of the group G that form a class by their own.

⋮

The second stage to construct the Character Table of a finite group G is to get the list of the character χ^k of its irreducible linear representations ρ_k. In the case of small enough groups the already established theorems may be enough to find them all. We recall the elementary property $\chi^k(e) = d_k$, the equations $\sum_k d_k^2 = n_{\mathrm{G}}$ and $\sum_k d_k \chi^k(g \neq e) = 0$ inferred from the regular representation $\rho_{\mathrm{G}} = \sum_k d_k \rho_k$, the equality $d_{\mathbb{C}[\mathfrak{C}_{\mathrm{G}}]} = n_{\mathfrak{C}}$ between the total number of the χ^k and that of the conjugacy classes $\mathfrak{C}_i$ and, of course, the orthonormality of the χ^k. Denoting χ_i^k the value of the character χ^k of an irreducible representation $\rho_k : \mathrm{G} \rightarrow \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$ over a conjugacy class $\mathfrak{C}_i$, the first orthonormality equation (3.3) re-writes:

$$\sum_i n_{\mathfrak{C}_i} (\chi_i^q)^\star \chi_i^k = n_{\mathrm{G}} \delta_{kd} \tag{3.17}$$

where $n_{\mathfrak{C}_i}$ is the number of elements in the conjugacy class $\mathfrak{C}_i$ and n_{G} the order of the group G. This makes up a "Row-by-Row Orthogonality Theorem" for the Character Table. The second orthonormality equation (3.16) re-writes:

$$\sum_k \chi_i^k (\chi_j^k)^\star = \frac{n_{\mathrm{G}}}{n_{\mathfrak{C}_i}} \delta_{ij} \tag{3.18}$$

which makes up a "Column-by-Column Orthogonality Theorem" for the Character Table. It finally may be remembered that, since G is a finite group, the character value χ_i^k is the sum of d_k terms each of which is an n_{g_i}-root of 1, the multiplicative unit of the complex numbers, where n_{g_i} is the order of the elements g_i of the class $\mathfrak{C}_i$.

Consider for purpose of illustration the geometric group of the rotations in the 3-dimensional space about the center of a tetrahedron that leaves the tetrahedron invariant. It is denoted G = 23 by the crystallographers and consists in 2-fold rotations about 3 distinct axes, that permute the summits by pairs, and 3-fold rotations about 4 distinct axes, that permute three summits circularly. The group, mathematically, is isomorphic to the group of even permutation of a set $\{a, b, c, d\}$ of 4 objects. It is recalled that a permutation π is even iff it decomposes itself into an even number of transpositions, that is to say iff its signature is $sign(\pi) = +1$. We have 3 elements of order 2: $\{g_x \equiv (ab)(cd)\}$, $\{g_y \equiv (ac)(bd)\}$, $\{g_z \equiv (ad)(bc)\}$ and 8 elements of order 3: $\{g_t \equiv (abc)\}$, $g_t g_x, \ldots$, $\{g_t^2 \equiv (acb)\}$, $g_t^2 g_x, \ldots$. With the unit e this corresponds to a group of order $n_{23} = 12$. One easily establishes that $g_t g_x g_t^{-1} = g_z$, $g_t g_z g_t^{-1} = g_y$, $g_t g_y g_t^{-1} = g_x$ and $g_t^2 (g_t g_x) g_t^{-2} = g_x g_t^{-2} = g_t g_y g_t^{-1} = g_t g_y, \ldots$, which leads to distinguish 4 conjugacy classes: $\mathfrak{C}_1 = \{e\}$, $\mathfrak{C}_2 = \{g_x, g_y, g_z\}$, $\mathfrak{C}_3 = \{g_t, g_t g_x, g_t g_y, g_t g_z\}$ and $\mathfrak{C}_4 = \{g_t^2, g_t^2 g_x, g_t^2 g_y, g_t^2 g_z\}$. We then must have 4 irreducible representations ρ_k $(k = 1, 4)$ with character χ^k $(k = 1, 4)$ and dimension d_k $(k = 1, 4)$. It is recalled that $d_k \neq 0$ $\forall k$ so the dimensional closure equation $\sum_k d_k^2 = n_{23}$ imposes that $d_1 = d_2 = d_3 = 1$ and $d_3 = 3$. One of the irreducible representations, ρ_1, necessarily is the trivial representation contained exactly once in the regular representation, whence $\chi_j^1 = 1$ $(j = 1, 4)$, which fills the first row of the Character Table. Since $\chi^k(e) = d_k$, we have $\chi_1^k = 1$ $(k = 1, 3)$ and $\chi_1^4 = 3$, which fills the first column of the Character Table. The other elements of the Character Table can be inferred from the orthogonality theorems for the character, keeping in mind that χ_2^k $(k = 2, 3)$ are square roots ± 1 of 1, χ_3^k $(k = 2, 3)$ and χ_4^k $(k = 2, 3)$ are cubic roots $\{1, \omega, \omega^\star\}$ of 1 with $\omega = \exp\left(\frac{2i\pi}{3}\right)$, χ_2^4 the sum of $d_3 = 3$ square roots of 1 and χ_j^4 $(j = 3, 4)$ the sum of $d_3 = 3$ cubic roots of 1. Considering the $\mathfrak{C}_1 - \mathfrak{C}_2$ column-by-column orthogonality we get $1 + \chi_2^2 + \chi_2^3 + 3\chi_2^4 = 0$, with $\chi_2^2 = \pm 1$ and $\chi_2^3 = \pm 1$. χ_2^4 *à priori* can take the values $-3, -1, 1, 3$, to which would correspond respectively the values $8, 2, -4, -10$ for $\chi_2^2 + \chi_2^3$. It follows that the only consistent values are $\chi_2^2 = 1, \chi_2^3 = 1$ and $\chi_2^4 = -1$, which fills the second column of the Character Table. The $\mathfrak{C}_1 - \mathfrak{C}_3$ and $\mathfrak{C}_2 - \mathfrak{C}_3$ column-by-column orthogonality then imposes that $1 + \chi_3^2 + \chi_3^3 = 0$ and $\chi_3^4 = 0$. One similarly has $1 + \chi_4^2 + \chi_4^3 = 0$ and $\chi_4^4 = 0$ by the $\mathfrak{C}_1 - \mathfrak{C}_4$ and $\mathfrak{C}_2 - \mathfrak{C}_4$ column-by-column orthogonality, which immediately fills the 4-th row. $1 + \chi_j^2 + \chi_j^3 = 0$ $(j = 3, 4)$ implies that if $\chi_3^2 = \omega$ then $\chi_3^3 = \omega^\star$ in which case $\chi_4^2 = \omega^\star$ then $\chi_4^3 = \omega$ by the $\mathfrak{C}_3 - \mathfrak{C}_4$ column-by-column orthogonality. We finally get the Character Table:

Table 1. Character Table of the Tetrahedron Group 23.

	$\mathfrak{C}_1 = \{e\}$	$\mathfrak{C}_2 = \{g_x, g_y, g_z\}$	$\mathfrak{C}_3 = \{g_t, g_t g_x, g_t g_y, g_t g_z\}$	$\mathfrak{C}_4 = \{g_t^2, g_t^2 g_x, g_t^2 g_y, g_t^2 g_z\}$
χ^1	1	1	1	1
χ^2	1	1	$\omega = \exp\left(\frac{2i\pi}{3}\right)$	$\omega^\star = \exp\left(\frac{4i\pi}{3}\right)$
χ^3	1	1	$\omega^\star = \exp\left(\frac{4i\pi}{3}\right)$	$\omega = \exp\left(\frac{2i\pi}{3}\right)$
χ^4	3	-1	0	0

The construction of the Character Table as above performed is rather unwieldy and reveals itself inefficient as the order n_{G} of the group G is increased. As a matter of fact, a number of additional theorems may be formulated that offer tools to forge powerful search algorithms, taking advantage of decompositions of groups into direct or semi-direct products of subgroups or else direct sums of subgroups, involving the concept of induced representation, making use of conjugacy class multiplication, exploiting arithmetic properties of the characters, ... A few of these theorems and methods will be approached in the following but only sketchily.

3.6 Projectors and exchangers

As to fully discern the effects of a symmetry group in the concrete instances it actually is inevitable to have to explicitly determine the invariant subspaces of the linear representations. One then is sent back to the discomforts of the arbitrariness associated with the intertwinings of the representations and of the consequent lack in general of a natural decomposition of a completely reducible linear representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$ of a group G into the irreducible representations $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$. This clearly prompts us to formulate a standard method, although not unique, of reduction.

An exception is the coarse-grained canonical decomposition $\rho = \bigoplus_k \xi_k$ of the linear representation ρ into isotypical components $\xi_k : \mathrm{G} \to \mathrm{GL}(\mathrm{X}_k, \mathbb{C})$, these being isomorphic to the direct sum of n_k copies of the irreducible representations ρ_k: $\xi_k = \bigoplus_s {\eta_k}^s$ and $\mathrm{X}_k = \bigoplus_s {\mathrm{V}_k}^s$ with ${\eta_k}^s \sim \rho_k$ and ${\mathrm{V}_k}^s \cong \mathrm{V}_k$ $(s = 1, n_k)$ or else, more symbolically, $\xi_k \sim n_k \rho_k$ and $\mathrm{X}_k \cong \mathrm{V}_k^{\oplus n_k}$. As proved from Schur 2 the canonical decomposition is unique, which implies that the isotypical components ξ_k can be unambiguously determined. ξ_k for each k is nothing but the restriction of ρ to the representation space X_k and only a little intuition is necessary to find out that each subspace X_k of the representation space V is fully identified by the linear operator on V given by the formula

$$\mathcal{P}_k = \frac{d_k}{n_\mathrm{G}} \sum_g (\chi^k(g))^\star \rho(g) \tag{3.19}$$

It indeed is inferred from the equation (3.13) that the restriction of $\mathcal{P}_k$ on every subspace ${\mathrm{V}_k}^s$ of V that is isomorphic to the representation space V_k of the irreducible representation ρ_k is the identity operator $1_{{\mathrm{V}_k}^s \cong \mathrm{V}_k}$ and the zero operator on any other subspace of V. A linear operator the restriction of which on a family of spaces is the identity (resp. zero) operator is the identity (resp. zero) operator on the direct sum space of the family, symbolically $\bigoplus_s 1_{{\mathrm{V}_k}^s} = 1_{\bigoplus_s {\mathrm{V}_k}^s}$ (resp. $\bigoplus_s 0_{{\mathrm{V}_k}^s} = 0_{\bigoplus_s {\mathrm{V}_k}^s}$). It follows that $\mathcal{P}_k$ is the identity operator on the representation space $\mathrm{X}_k = \bigoplus_s {\mathrm{V}_k}^s$ of the isotypical component ξ_k and the zero operator everywhere else in the representation space V, that is to say **$\boldsymbol{\mathcal{P}_k}$ is the projector of $\mathbf{V} = \boldsymbol{\bigoplus_q \mathrm{X}_q}$ onto $\mathbf{X}_{\boldsymbol{k}}$**.

Consequently, to formulate a method for a standard reduction of any linear representation ρ of a group G, it suffices to do so for each of its isotypical components ξ_k. Choose, in that purpose, a basis $\{\hat{e}_n\}_{n=1,\ldots,d_k}$ in the representation space V_k of each irreducible representation ρ_k of G and denote $\Gamma^k : \mathrm{G} \to \mathrm{GL}(\mathrm{d}_k, \mathbb{C})$ the matrix representation associated with ρ_k with respect to the selected basis in each V_k. We are free to define for each k the linear operators

$$\mathcal{Q}^k_{mn} = \frac{d_k}{n_\mathrm{G}} \sum_g \Gamma^k_{nm}(g^{-1}) \rho(g) \quad (m,\ n) \in \{1,\ 2,\ \cdots,\ d_k\}^2 \tag{3.20}$$

on the representation space V of ρ. As from the orthogonality theorem for the matrix representations, to be precise from the equation (2.34), it immediately is inferred that $\forall (n, m)$ $\mathcal{Q}^k_{mn}$ is null on every subspace $\mathrm{V}^s_{q \neq k}$ and therefore on every subspace $\mathrm{X}_{q \neq k} = \bigoplus_s \mathrm{V}^s_{q \neq k}$ of V. One similarly establishes, focussing solely at X_k, that if $\{\hat{e}^s_n\}_{n=1,\ldots,d_k}$ in ${\mathrm{V}_k}^s$ stands for an isomorphic replica of $\{\hat{e}_n\}_{n=1,\ldots,d_k}$ in V_k then

$$\mathcal{Q}^k_{mn}(\hat{e}^s_r) = \begin{cases} \hat{e}^s_m & \text{if } n = r \\ 0 & \text{otherwise} \end{cases} \tag{3.21}$$

It thus is found out that **$\boldsymbol{\mathcal{Q}^k_{mm}}$ projects the representation space V onto** $im(\mathcal{Q}^k_{mm}) = Span(\hat{e}^1_m, \ldots, \hat{e}^s_m, \ldots) = \mathbf{X}^{\boldsymbol{m}}_{\boldsymbol{k}} \subset \mathbf{X}_{\boldsymbol{k}}$ and that $\sum_m \mathcal{Q}^k_{mm} = \mathcal{P}^k$ so that $\mathbf{X}_{\boldsymbol{k}} = \boldsymbol{\bigoplus_m} \mathbf{X}^{\boldsymbol{m}}_{\boldsymbol{k}}$. One also deduces that $\mathcal{Q}^k_{mn}$ defines an isomorphism of X^m_k to X^n_k and is null elsewhere in the space V, that is to say $\boldsymbol{\mathcal{Q}^k_{mn}}$

transforms $\mathbf{X}_k^m$ into $\mathbf{X}_k^n$. It further is shown through the equation (3.21) that $\mathcal{Q}_{mn}^k \circ \mathcal{Q}_{rt}^k = \mathcal{Q}_{mt}^k$ if $n = r$ and zero otherwise and that $\rho(g) \circ \mathcal{Q}_{mr}^k = \sum_n \Gamma_{nm}^k \mathcal{Q}_{nr}^k$. It follows that if $\vec{x}_k^1 \in \mathrm{X}_k^1$ is a non null vector then the vectors $\vec{x}_k^m = \mathcal{Q}_{m1}^k(\vec{x}_k^1) \in \mathrm{X}_k^m$ are linearly independent and makes up a basis of a G-invariant subspace $\mathrm{V}_k(\vec{x}_k^1)$ of dimension d_k, which is isomorphic to V_k. Choosing a basis $\{\vec{x}_k^1, \ldots \vec{x}_k^s, \ldots\}$ in X_k^1, one gets a collection of subspaces $\mathrm{V}_k(\vec{x}_k^1), \ldots \mathrm{V}_k(\vec{x}_k^s), \ldots$ the direct sum of which gives back X_k. It then is clear that the restrictions of ρ to these G-invariant subspaces can be taken as the ρ_k-copy components of the searched standard decomposition of the isotypical component ξ_k. One may proceed systematically in the concrete cases, by selecting an arbitrary basis in the representation space V of ρ and projects each vector of this basis onto the spaces X_k^m by using the **projectors** $\mathcal{Q}_{mm}^k$ then applies the **exchangers** $\mathcal{Q}_{mn}^k$ to get the bases of all the standard G-invariant subspaces.

Generalization to the fields $\mathbb{K}$ whose characteristic $char(\mathbb{K})$ does not divide the order n_{G} of the group G is straightforward as well as to the compact groups G. In the latter case the projectors and exchangers are built by replacing the summation $\frac{1}{n_{\mathrm{G}}} \sum_g$ by the Haar integration: $\mathcal{P}_k = d_k \int_{\mathrm{G}} (\chi^k(g))^\star \rho(g)\, dg$ and $\mathcal{Q}_{mn}^k = d_k \int_{\mathrm{G}} \Gamma_{nm}^k(g^{-1}) \rho(g)\, dg$ with $(m,\ n) \in \{1,\ 2,\ \ldots,\ d_k\}^2$.

4. MISCELLANEA

A few additional topics are more succinctly discussed in this section, in order to only catch a glimpse of the wealth of the topic. Constructions of new linear representations of groups from existing representations through tensor products of the representation spaces or through groups products are described. The concept of induced representation is approached with a qualitative discussion of a few essential theorems. A method of systematic search of the irreducible representations of finite groups is mentioned. The section ends with a very short description of group representations on more general mathematical objects than vector spaces.

4.1 Tensor product

A vector space V over a field $\mathbb{K}$ is the **tensor product** $\mathrm{V}_1 \otimes \mathrm{V}_2$ **of two vector spaces** V_1 and V_2 over the field $\mathbb{K}$ iff it is endowed with an application $(\vec{v}_1 \in \mathrm{V}_1, \vec{v}_2 \in \mathrm{V}_2) \mapsto \vec{v} = \vec{v}_1 \otimes \vec{v}_2 \in \mathrm{V}$ linear in each of the two variables $\vec{v}_1$ and $\vec{v}_2$. It is shown that V is unique up to isomorphism. If $\{\hat{e}_{n_i}^i\}_{n_i=1,\ldots,d_i}$ $(i = 1, 2)$ is a basis of V_i $(i = 1, 2)$ then $\{\hat{e}_{n_1}^1 \otimes \hat{e}_{n_2}^2\}_{n_1=1,\ldots,d_1, n_2=1,\ldots,d_2}$ makes up a basis of V: the **dimension** of $\mathrm{V}_1 \otimes \mathrm{V}_2$ is the product $\boldsymbol{d = d_1 d_2}$ of the dimensions of V_1 and V_2. The tensor product of vector spaces is associative and distributive with respect to the direct sum, to be precise $\mathrm{U} \otimes (\mathrm{V} \otimes \mathrm{W}) \cong (\mathrm{U} \otimes \mathrm{V}) \otimes \mathrm{W}$ and $(\mathrm{U} \oplus \mathrm{V}) \otimes \mathrm{W} \cong (\mathrm{U} \otimes \mathrm{W}) \oplus (\mathrm{V} \otimes \mathrm{W})$ are natural isomorphisms. Natural is to mean that no choice of basis is requested to produce the property. Let α_i $(i = 1, 2)$ be a linear operator on the vector space V_i $(i = 1, 2)$. The **tensor product** $\alpha_1 \otimes \alpha_2$ **of the linear operators** α_1 and α_2 is the linear operator on the tensor product vector space $\mathrm{V}_1 \otimes \mathrm{V}_2$ defined as $(\alpha_1 \otimes \alpha_2)(\vec{v}_1 \otimes \vec{v}_2) = \alpha_1(\vec{v}_1) \otimes \alpha_2(\vec{v}_2)$ $\forall (\vec{v}_1 \in \mathrm{V}_1, \vec{v}_2 \in \mathrm{V}_2)$. If A^i $(i = 1, 2)$ is the matrix representative of α_i $(i = 1, 2)$ with respect to the basis $\{\hat{e}_{n_i}^i\}_{n_i=1,\ldots,d_i}$ $(i = 1, 2)$ in the vector space V_i $(i = 1, 2)$ then the **matrix representative** of $\alpha_1 \otimes \alpha_2$ with respect to the basis $\{\hat{e}_{n_1}^1 \otimes \hat{e}_{n_2}^2\}_{n_1=1,\ldots,d_1, n_2=1,\ldots,d_2}$ in the vector space $\mathrm{V}_1 \otimes \mathrm{V}_2$ is the matrix $\mathrm{A}^1 \otimes \mathrm{A}^2$ whose entries are given in terms of the entries of the matrices A^i $(i = 1, 2)$ as

$$(\mathrm{A}^1 \otimes \mathrm{A}^2)_{(i,k)(j,l)} = \mathrm{A}_{ij}^1 \mathrm{A}_{kl}^2 \tag{4.1}$$

which is checked by observing that the application of $(\mathrm{A}^1 \otimes \mathrm{A}^2)$ to the basis vector $\hat{e}_j^1 \otimes \hat{e}_l^2$ contains the basis vector $\hat{e}_i^1 \otimes \hat{e}_k^2$ with the awaited coefficient $\mathrm{A}_{ij}^1 \mathrm{A}_{kl}^2$. An interesting property is

$$\mathrm{Tr}(\mathrm{A}^1 \otimes \mathrm{A}^2) = \sum_{i,k} (\mathrm{A}^1 \otimes \mathrm{A}^2)_{(i,k)(i,k)} = \sum_{i,k} \mathrm{A}_{i,i}^1 \mathrm{A}_{k,k}^2 = \sum_i \mathrm{A}_{i,i}^1 \sum_k \mathrm{A}_{k,k}^2 = \mathrm{Tr}(\mathrm{A}^1)\mathrm{Tr}(\mathrm{A}^2) \tag{4.2}$$

If the operators α_i $(i = 1, 2)$ are diagonalizable then so is $A^1 \otimes A^2$ and if $\{\hat{e}^i_{n_i}\}_{n_i=1,\ldots,d_i}$ $(i = 1, 2)$ are the eigenbasis of α_i $(i = 1, 2)$ with eigenvalues $\lambda^i_{n_i}$ $(n_i = 1, \ldots, d_i)$ $(i = 1, 2)$ then so is $\{\hat{e}^1_{n_1} \otimes \hat{e}^2_{n_2}\}_{n_1=1,\ldots,d_1,n_2=1,\ldots,d_2}$ with eigenvalues $\lambda^1_{n_1}\lambda^2_{n_2}$ $(n_1 = 1, \ldots, d_1, n_2 = 1, \ldots, d_2)$. It then follows that $\mathrm{Det}(A^1 \otimes A^2) = \{\mathrm{Det}(A^1)\}^{d_2}\{\mathrm{Det}(A^2)\}^{d_1}$.

Now let $\rho_1 : G \to GL(V_1, \mathbb{C})$ and $\rho_2 : G \to GL(V_2, \mathbb{C})$ be two linear representations of the group G. The **tensor product** $\rho = \rho_1 \otimes \rho_2$ **of the linear representations** ρ_1 and ρ_2 is the linear representation $\rho : G \to GL(V, \mathbb{C})$ that associates to each g in G the linear operator $\rho(g)$ on the tensor product vector space $V = V_1 \otimes V_2$ such that $\rho(g)(\vec{v}_1 \otimes \vec{v}_2) = \rho_1(g)(\vec{v}_1) \otimes \rho_2(g)(\vec{v}_2)$, $\forall \vec{v}_1 \in V_1\ \forall \vec{v}_2 \in V_2$. ρ is uniquely defined up to isomorphism. The matrix representative $\Gamma(g)$ of the linear operator $\rho(g)$ for each g in G with respect to the basis $\{\hat{e}^1_{n_1} \otimes \hat{e}^2_{n_2}\}_{n_1=1,\ldots,d_1,n_2=1,\ldots,d_2}$ is the tensor product $\Gamma^1(g) \otimes \Gamma^2(g)$ of the matrix representatives $\Gamma^i(g)$ $(i = 1, 2)$ of the linear operators $\rho_i(g)$ $(i = 1, 2)$ with respect to the bases $\{\hat{e}^i_{n_i}\}_{n_i=1,\ldots,d_i}$ $(i = 1, 2)$ in the vector spaces V_i $(i = 1, 2)$:

$$\Gamma(g) = \Gamma^1(g) \otimes \Gamma^2(g) \equiv \begin{pmatrix} \Gamma^1_{11}(g)\ \left(\Gamma^2(g)\right) & \cdots & \Gamma^1_{1d_1}(g)\ \left(\Gamma^2(g)\right) \\ \vdots & & \vdots \\ \Gamma^1_{d_1 1}(g)\ \left(\Gamma^2(g)\right) & \cdots & \Gamma^1_{d_1 d_1}(g)\ \left(\Gamma^2(g)\right) \end{pmatrix} \quad \forall g \in G \tag{4.3}$$

One says that the matrix representation Γ is the **tensor product of the matrix representations** Γ^1 **and** Γ^2, symbolically $\Gamma = \Gamma^1 \otimes \Gamma^2$. Generalization to multiple tensor product is obvious. Consider then a linear representation $\rho : G \to GL(V, \mathbb{C})$ of the group G. The **ν-th tensor power** of the vector space V is the vector space $V^{\otimes\nu} = V \otimes \ldots \otimes V$ (ν times) and the ν-th tensor power of the linear representation ρ is the linear representation $\rho^{\otimes\nu} : G \to GL(V^{\otimes\nu}, \mathbb{C})$ that associates to each g in G the linear operator $\rho^{\otimes\nu}(g) = \rho(g) \otimes \ldots \otimes \rho(g)$ (ν times) on $V^{\otimes\nu}$. If $\{\hat{e}_n\}_{n=1,\ldots,d}$ is a basis of V then a basis in $V^{\otimes\nu}$ is obtained from the collection of vectors $\hat{e}_{n1} \otimes \ldots \otimes \hat{e}_{n\nu}$ where the indices $n1, \ldots, n\nu$ range over $\{1, \ldots, d\}^\nu$: the dimension of $\rho^{\otimes\nu}$ is d^ν. Applying $\rho^{\otimes\nu}(g)$ before or after any permutation $\pi : \hat{e}_{n1} \otimes \ldots \otimes \hat{e}_{n\nu} \to \hat{e}_{\pi(n1)} \otimes \ldots \otimes \hat{e}_{\pi(n\nu)}$ of factors leads to the same result. This means that the action of the group $\mathcal{S}_\nu$ of permutations π commutes with $\rho^{\otimes\nu}$. $\mathcal{S}_\nu$ thus must preserves the canonical decomposition of $\rho^{\otimes\nu}$. So every $\mathcal{S}_\nu$-isotypical component of $\rho^{\otimes\nu}$ makes up a sub-representation of G. Among these it is customary to discern the ν-th symmetric power $\rho_{\mathrm{Sym}^\nu} : G \to GL(\mathrm{Sym}^\nu V, \mathbb{C})$ associated with the trivial representation of $\mathcal{S}_\nu$ and the ν-th alternate power $\rho_{\mathrm{Alt}^\nu} : G \to GL(\mathrm{Alt}^\nu V, \mathbb{C})$ associated with the sign representation of $\mathcal{S}_\nu$, which is defined by declaring that every transposition produces a multiplication by -1. Define the linear operators $\Pi_\pm : \vec{v}_1 \otimes \ldots \otimes \vec{v}_\nu \mapsto \frac{1}{n!}\sum_{\pi\in\mathcal{S}_\nu}(\pm)^{\mathcal{N}(\pi)}\vec{v}_{\pi(1)} \otimes \ldots \otimes \vec{v}_{\pi(\nu)}$ on $V^{\otimes\nu}$, where $\mathcal{N}(\pi)$ is the number of transposition under which π decomposes. One easily shows that Π_+ is a projector of $V^{\otimes\nu}$ onto $\mathrm{Sym}^\nu V$ and Π_- a projector of $V^{\otimes\nu}$ onto $\mathrm{Alt}^\nu V$. The vectors $\Pi_+(\hat{e}_{n1} \otimes \ldots \otimes \hat{e}_{n\nu})$ $(1 \le n_1 \le \ldots n\nu \le d)$ make up a basis of $\mathrm{Sym}^\nu V$ and the vectors $\Pi_-(\hat{e}_{n1} \otimes \ldots \otimes \hat{e}_{n\nu})(1 \le n_1 < \ldots n\nu < d)$ a basis of $\mathrm{Alt}^\nu V$. If $\nu = 2$ then one gets the **symmetric square** ρ_{Sym^2} **and the alternate square** ρ_{Alt^2}. Note that $\rho \otimes \rho = \rho_{\mathrm{Sym}^2} \oplus \rho_{\mathrm{Alt}^2}$. The dimension of ρ_{Sym^2} is $d_{\mathrm{Sym}^2} = d(d+1)/2$ and the dimension of ρ_{Alt^2} is $d_{\mathrm{Alt}^2} = d(d-1)/2$. The matrix representation associated with ρ_{Sym^2} with respect to the symmetrized basis $\{\hat{e}_{n1} \otimes \hat{e}_{n2} + \hat{e}_{n2} \otimes \hat{e}_{n1}\}_{1\le n1\le n2\le d}$ defines the **symmetric square matrix representation** $[\Gamma \otimes \Gamma]$ and the matrix representation associated with ρ_{Alt^2} with respect to the antisymmetrized basis $\{\hat{e}_{n1} \otimes \hat{e}_{n2} - \hat{e}_{n2} \otimes \hat{e}_{n1}\}_{1\le n1<n2\le d}$ defines the **antisymmetric square matrix representation** $\{\Gamma \otimes \Gamma\}$. Of course $\Gamma \otimes \Gamma = [\Gamma \otimes \Gamma] \oplus \{\Gamma \otimes \Gamma\}$.

The characters of the tensor products of linear representations are elementarily determined:

- The character of $\boldsymbol{\Gamma = \bigotimes_i \Gamma^i}$ is $\boldsymbol{\chi = \prod_i \chi^i}$, where χ^i stands for the character of Γ^i. Evident from the property $\mathrm{Tr}\,[A \otimes B] = \mathrm{Tr}\,[A]\,\mathrm{Tr}\,[B]$ for any pair of matrices A and B.
- The character of the symmetric square $\boldsymbol{[\Gamma \otimes \Gamma]}$ of Γ with character χ is determined as $\boldsymbol{\forall g \in G,\ \chi^2_S(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)}$. Denoting $\epsilon_1, \ldots, \epsilon_d$ the eigenvalues of $\Gamma(g)$, one indeed computes $\chi^2_S(g) = \sum_{i\le j}\epsilon_i\epsilon_j = \sum_i \epsilon_i^2 + \sum_{i<j}\epsilon_i\epsilon_j = \sum_i \epsilon_i^2 + \frac{1}{2}\left((\sum_i \epsilon_i)^2 - \sum_i \epsilon_i^2\right) = \frac{1}{2}((\sum_i \epsilon_i)^2 + \sum_i \epsilon_i^2) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$.

- The character of the antisymmetric square **$\{\Gamma \otimes \Gamma\}$** of Γ with character χ is determined as **$\forall g \in G,\ \chi_A^2(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$**. Denoting $\epsilon_1, \ldots, \epsilon_d$ the eigenvalues of $\Gamma(g)$, one indeed computes $\chi_A^2(g) = \sum_{i<j} \epsilon_i \epsilon_j = \frac{1}{2}\left((\sum_i \epsilon_i)^2 - \sum_i \epsilon_i^2\right) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$. Note the equality $\chi^2 = \chi_S^2 + \chi_A^2$, which reflects the fact that $\rho \otimes \rho = \rho_{\mathrm{Sym}^2} \oplus \rho_{\mathrm{Alt}^2}$.

$\vdots$

A tensor product $\rho = \rho_k \otimes \rho_q$ of two irreducible representations ρ_k and ρ_q of a group G generally is not irreducible. Its decomposition into irreducible components ρ_t standardly writes

$$\rho_k \otimes \rho_q \sim \bigoplus_t n_{kq}{}^t\ \rho_t \tag{4.4}$$

where the multiplicity coefficients $n_{kq}{}^t$ are generically called **Clebsh-Gordan** coefficients. Using the equations 3.3 and 3.4, these easily are computed as

$$n_{kq}{}^t = \langle \chi^t \mid \chi^k\ \chi^q \rangle = \frac{1}{n_G} \sum_{g \in G} (\chi^t(g))^\star \chi^k(g) \chi^q(g) \tag{4.5}$$

It immediately is inferred by comparison with equation (3.3) that if $\chi^t(g) = 1\ \forall g \in G$ then $(\chi^k)^\star = \chi^q$. In other words the trivial representation of a group G is contained once and only once in the reduction of the tensor product $\rho_k \otimes \rho_q$ of any two irreducible representations ρ_k and ρ_q of G iff these are either complex conjugate, $\rho_k \sim \rho_q^\star$, or adjoint of each other, $\rho_k \sim \rho_q^\dagger$.

4.2 Group products

The **direct product** $G_1 \times G_2$ of two groups G_1 and G_2 by definition is the group formed by endowing the set $\{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$ with the composition law

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2) \quad \forall g_1, h_1 \in G_1 \text{ and } \forall g_2, h_2 \in G_2 \tag{4.6}$$

If G_i $(i = 1, 2)$ is of order n_{G_i} $(i = 1, 2)$ then the order of $G_1 \times G_2$ is $n_{G_1 \times G_2} = n_{G_1} n_{G_2}$. The group G_1 is isomorphic to the subgroup $G_1 \times E_2$ of the group $G_1 \times G_2$ consisting in the pairs (g_1, e_2) where e_2 is the unit element of G_2. It thus can be identified with it. The group G_2 similarly can be identified with the subgroup $E_1 \times G_2$ of the group $G_1 \times G_2$ consisting in the pairs (e_1, g_2) where e_1 is the unit element of G_1. Each element of $G_1 \times E_2$ obviously commutes with each element of $E_1 \times G_2$. Conversely, let G be a group containing G_1 and G_2 as subgroups such that i- every g in G writes uniquely as $g = g_1 g_2$ with g_1 in G_1 and g_2 in G_2, ii- for $g_1 \in G_1$ and $g_2 \in G_2$ one has $g_1 g_2 = g_2 g_1$. The product of two elements $g = g_1 g_2$ and $h = h_1 h_2$ can then be written as $gh = g_1 g_2 h_1 h_2 = (g_1 h_1)(g_2 h_2)$. If we let $(g_1, g_2) \in G_1 \times G_2$ correspond to the element $g_1 g_2 \in G$ we then obtain an isomorphism of $G_1 \times G_2$ onto G. In this case, G is identified with $G_1 \times G_2$ and one says that G is the direct product of its subgroups G_1 and G_2.

Now let $\rho_i : G_i \to GL(V_i, \mathbb{C})$ $(i = 1, 2)$ be linear representations of the group G_i $(i = 1, 2)$. We may define a linear representation $\rho = \rho_1 \otimes \rho_2$ of the group product $G_1 \times G_2$ on the tensor product vector space $V = V_1 \otimes V_2$ by setting

$$(\rho_1 \otimes \rho_2)(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2) \quad \forall g_1 \in G_1\ \forall g_2 \in G_2 \tag{4.7}$$

$\rho_1 \otimes \rho_2$ is unique up to isomorphism and is called the **tensor product of the representations** ρ_1 and ρ_2. If χ_i is the character of ρ_i $(i = 1, 2)$ then the character χ of $\rho = \rho_1 \otimes \rho_2$ is given by $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)\ \forall g_1 \in G_1\ \forall g_2 \in G_2$. If $\rho_1^k : G_1 \to GL(V_1, \mathbb{C})$ and $\rho_2^q : G_2 \to GL(V_2, \mathbb{C})$ are irreducible representations then $\langle \chi_1^k \mid \chi_1^k \rangle = 1$ and $\langle \chi_2^q \mid \chi_2^q \rangle = 1$ so that $\langle \chi_1^k \chi_2^q \mid \chi_1^k \chi_2^q \rangle = \langle \chi_1^k \mid \chi_1^k \rangle \langle \chi_2^q \mid \chi_2^q \rangle = 1$, which means that $\rho_1^k \otimes \rho_2^q$ is an irreducible representation of $G_1 \times G_2$. Assume now that φ is a class function on $G_1 \times G_2$, which is orthogonal to all the characters of the form $\chi_1^k \chi_2^q$, namely $\langle \varphi \mid \chi_1^k \chi_2^q \rangle = \sum_{(g_1, g_2)} (\varphi(g_1, g_2))^\star \chi_1^k(g_1) \chi_2^q(g_2) = 0$. Fixing χ_1^k and putting $\psi(g_2) = \sum_{g_1} \varphi(g_1, g_2)(\chi_1^k(g_2))^\star$ we get

$\sum_{g_2}(\psi(g_2))^\star\chi_2^q(g_2) = 0\ \forall\chi_2^q$. ψ is a class function so it is null. Since the same is true of every χ_1^k we conclude that φ is identically null on $G_1 \times G_2$. In order words, each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation $\rho_1^k \otimes \rho_2^q$, where ρ_1^k is an irreducible representation of G_1 and ρ_2^q an irreducible representation of G_2. Obviously these properties allow completely reducing the study of the representations of the group $G_1 \times G_2$ to that of the representation of the groups G_i ($i = 1, 2$).

Given two groups G and H and a morphism ψ of the group H into the group Aut(G) of the automorphisms of G. The **semi-direct product $\mathbf{G \rtimes_\psi H}$** of G and H with respect to the action ψ of H on G designates the set $\{(g, h) \mid g \in G, h \in H\}$ endowed with the composition law

$$(g, h)(g', h') = (g\ \psi(h)(g'), h\ h') \quad \forall g, g' \in G \text{ and } \forall h, h' \in H \tag{4.8}$$

It is almost obvious that $G \rtimes_\psi H$ shows a group structure. One has the so-called exact sequence $1 \to G \to G \rtimes_\psi H \to H \to 1$, with the injective homomorphism $\mathfrak{I} : G \to G \rtimes_\psi H$ defined by $\mathfrak{I}(g) = (g, e_H)$, the surjective homomorphism $\mathfrak{S} : G \rtimes_\psi H \to H$ defined by $\mathfrak{S}(g, h) = h$ and, as it is required for an exact sequence, $ker(\mathfrak{S}) = im(\mathfrak{I})$. The subgroup $\mathfrak{I}(G) = G \times \{e_H\}$ is normal. It is observed that by identifying G with $G \times \{e_H\}$ and H with $\{e_G\} \times H$ every semi-direct product K can be conceived as a semi-direct product of two subgroups G and H associated with the morphism ψ of K into Aut(G) defined by $\psi(h)(g) = hgh^{-1}$ with $h \in H$. Conversely if K is an extension of G by H, that is if we have the sequence $1 \to G \to K \to H \to 1$ with $\mathfrak{I} : G \to K$ injective, $\mathfrak{S} : K \to H$ surjective and $ker(\mathfrak{S}) = im(\mathfrak{I})$ (exact sequence), and if K contains a subgroup L on which the restriction of $\mathfrak{S}$ is an isomorphism to H then K is isomorphic to the semi-direct product $\mathfrak{I}(G) \rtimes_\psi L$ with ψ the morphism of K into Aut(G) defined by $\psi(h)(g) = hgh^{-1}$ with $h \in H$. K can be conceived also as isomorphic to a semi-direct product $M \rtimes L$ with each element k in K writing uniquely as gh with g in a subgroup M isomorphic to G and h in a subgroup L isomorphic to H. It is clear that identifying a finite group as a semi-direct product should be useful to the study of its (irreducible) representations.

Consider for purpose of illustration the Tetrahedron Group 23 discussed in Section 3.5. A simple inspection of the composition of its elements suggests that it contains as subgroups the group M formed by the set $\{e, g_x, g_y, g_y\}$ and the group L formed by the set $\{e, g_t, g_t^2\}$. M is normal but L is not. $M \cap L = \{e\}$ and every element of 23 writes uniquely as gh with $g \in M$ and $h \in L$. It follows that $23 = M \rtimes L$. Note that 23 is not a direct product because M and L do not commute. Another reason is that L is not normal. Now, the group L is isomorphic to the cyclic group $\mathcal{C}_3$, which is abelian. The characters χ^i ($i = 1, 3$) of its irreducible representations, which are 1-dimensional, extends immediately to 23 as $\chi^i(gh) = \chi^i(h)$ ($i = 1, 3$) for $g \in M$ and $h \in L$. The last character χ^4 of the group 23 then may be obtaind by using for instance the row-by-row orthogonality theorem for the characters, to be precise the equation (3.17).

When, more generally, a group G is the semi-direct product $G = A \rtimes H$ of a group H with an abelian group A then its irreducible representations are all obtained by the so-called method of **"little groups"**. The characters of the irreducible representations of the abelian group A, which are 1-dimensional, form a group $X = \mathrm{Hom}(A, \mathbb{C}^\star)$. G acts on X by $(g\chi)(a) = \chi(g^{-1}ag)$ for $g \in G$, $\chi \in X$ and $a \in A$ (cf. Section 4.5). Let $(\chi_i)_{i \in X/H}$ be a system of representatives for the orbits of H in X. For each $i \in X/H$, let H_i be the subgroup of H consisting of those elements h s.t. $h\chi_i = \chi_i$ and let $G_i = A \rtimes H_i$ be the corresponding subgroup of G. Extend the function χ_i to G_i by setting $\chi_i(ah) = \chi_i(a)$ for $a \in A$ and $h \in H_i$. Using the fact $h\chi_i = \chi_i\ \forall h \in H_i$ one finds that χ_i is a character of a 1-dimensional representation of G_i. Let ρ be an irreducible representation of H_i. Composing ρ with the canonical projection $G_i \to H_i$ gives an irreducible representation $\widetilde{\rho}$ of G_i. By tensor product of χ_i and $\widetilde{\rho}$ an irreducible representation $\chi_i \otimes \widetilde{\rho}$ of G_i is produced. Let $\theta_{i,\rho}$ be the corresponding induced representation of G (cf. Section 4.3). It then is shown that i- $\theta_{i,\rho}$ is irreducible (by the Mackey criterion), ii- if $\theta_{i,\rho} \sim \theta_{i',\rho'}$ then $i = i'$ and $\rho \sim \rho'$, iii- every irreducible representation of G is isomorphic to one of the $\theta_{i,\rho}$.

4.3 Induced representations

Let H be a subgroup of a group G and let $g \in$ G. It is recalled that the set $\mathbf{gH = \{gh \mid h \in H\}}$ by definition is the **left coset modulo H** containing the element g of G. Two elements h and f of G are said congruent modulo H if they belong to the same left coset: $\forall h \in \mathrm{G}\, \forall f \in \mathrm{G}\, \{h \in g\mathrm{H} \wedge f \in g\mathrm{H}\} \Leftrightarrow hf^{-1} \in \mathrm{H} \Leftrightarrow h \equiv f \pmod{H}$. Any two left cosets are either disjoint or identical. The **set of left cosets of H**, denoted **G/H**, makes up a **partition of G**. If n_{G} is the order of G and n_{H} the order of H then the number of elements of the set G/H defines the index $[\mathrm{G}:\mathrm{H}] = n_{\mathrm{G}}/n_{\mathrm{H}}$ of H in G. If an element g_L is chosen in each distinct left coset then one gets a subset $\mathbf{R = \{g_1, g_2, \ldots, g_{[G:H]}\}}$ of G called a **system of representatives of G/H**:

$$\mathrm{G} = g_1\mathrm{H} + g_2\mathrm{H} + \cdots + g_{[\mathrm{G:H}]}\mathrm{H} \tag{4.9}$$

Each g in G writes uniquely as $g = g_L h$, where $h \in$ H and $g_L \in$ R is a coset representative.

Now let $\boldsymbol{\rho} : \mathbf{G \to GL(V, \mathbb{C})}$ be a representation of the group G on the vector space V and $\boldsymbol{\theta} : \mathbf{H \to GL(W, \mathbb{C})}$ a representation of the **subgroup H of G** on a **subspace W of V**. If θ is a subrepresentation of the restriction $\boldsymbol{\xi} : \mathbf{H \to GL(V, \mathbb{C})}$, $\boldsymbol{h \mapsto \xi(h) = \rho(h)}$ of ρ to the subgroup H then the subspace $\rho(g)$W depends only on the left coset gH. Indeed, if g is replaced by gh with $h \in$ H then $\rho(gh)\mathrm{W} = \rho(g) \circ \rho(h)\mathrm{W} = \rho(g) \circ \xi(h)\mathrm{W} = \rho(g) \circ \theta(h)\mathrm{W} = \rho(g)\mathrm{W}$, because W is H-invariant through θ. We thus define a subspace W_Σ for each left coset Σ in the set G/H, which is a replica of W in V. By definition, **the representation ρ of the group G is induced by the subrepresentation θ of the restriction ξ of ρ to the subgroup H of G iff**

$$\mathrm{V} = \bigoplus_{\Sigma \in \mathrm{G/H}} \mathrm{W}_\Sigma = \bigoplus_{g_L \in \mathrm{R}} g_L \mathrm{W} \tag{4.10}$$

It immediately is deduced that: i- If d_{W} is the dimension of θ then the dimension of the induced representation ρ is $\mathbf{d_V = [G:H]\, d_W}$. ii- If ρ_1 is induced by θ_1 and if ρ_2 is induced by θ_2, then $\boldsymbol{\rho_1 \oplus \rho_2}$ **is induced by** $\boldsymbol{\theta_1 \oplus \theta_2}$. iii- The **regular representation** ρ_{G} **of G is induced by the regular representation** ρ_{H} **of H**. V then has a basis $\{\hat{e}_g\}_{g \in \mathrm{G}}$ indexed by G. So it suffices to take for W the subspace of V spanned by the set of vectors indexed by the subgroup H of G that is $\{\hat{e}_h\}_{h \in \mathrm{H}}$. iv- If ρ is induced by θ and **if** W_1 **is an H-invariant subspace of W** then the subspace $\mathbf{V_1 = \oplus_{g_L \in R} W_1}$ **is stable under G** and the representation of G on V_1 is induced by the representation of H in W_1. Using some of these properties one proves the existence of the induced representations. We indeed may assume, from property (ii), that θ is irreducible, in which case θ is isomorphic to a subrepresentation of the regular representation ρ_{H} of H, which, according to property (iii), can be induced to the regular representation ρ_{G} of G. Applying property (iv), we conclude that θ itself may be induced. Assume that θ induces another representation $\rho' : \mathrm{G} \to \mathrm{GL}(\mathrm{V}', \mathbb{C})$. Whatever the linear operator $\phi : \mathrm{W} \to \mathrm{V}'$ we are free to extend it to the linear operator $\Phi : \mathrm{V} \to \mathrm{V}'$ by putting $\Phi = \rho'(g_L) \circ \phi \circ (\rho(g_L))^{-1}$ on each replica g_LW of W. Φ does not depend on the choice of the left coset representative g_L and is well defined since V is the direct sum of the replicas g_LW. Φ obviously is unique. Observe that $\Phi \circ \rho(g) = \phi \circ \rho(g)\ \forall g \in$ G. If ϕ is the injection of W into V′ then Φ is the identity on W and satisfies $\Phi \circ \rho(g) = \rho'(g) \circ \Phi\ \forall g \in$ G so that $im(\Phi) \supset \rho'(g)\mathrm{W}\ \forall g \in$ G whence $im(\Phi) \supset \mathrm{V}'$. Since V′ and V have the same dimension $d_{\mathrm{V}} = [\mathrm{G}:\mathrm{H}]\, d_{\mathrm{W}}$, we observe that Φ is an isomorphism. As a consequence, ***for every representation*** $\boldsymbol{\theta} : \mathbf{H \to GL(W, \mathbb{C})}$ ***of a subgroup H of a group G on a subspace W of a vector space V there exists a representation*** $\boldsymbol{\rho} : \mathbf{G \to GL(V, \mathbb{C})}$ ***induced by θ, which is unique up to isomorphism***.

Choose a basis $\{\hat{e}_n\}_{n=1,\ldots,d_{\mathrm{W}}}$ in W and let Γ_{H} be the matrix representation associated with the representation $\theta : \mathrm{H} \to \mathrm{GL}(\mathrm{W}, \mathbb{C})$ with respect to this basis. With the vectors $\hat{e}_{nL} = \rho(g_L)(\hat{e}_n)$ $(n = 1, \ldots, d_{\mathrm{W}})$ a basis is built up in each replica g_LW of W, whence, since $\mathrm{V} = \bigoplus_{g_L \in \mathrm{R}} g_L \mathrm{W}$, the set $\{\hat{e}_{nL}\}_{n=1,\ldots,d_{\mathrm{W}}, L=1,\ldots,[\mathrm{G:H}]}$ makes up a basis of V. $\forall g \in \mathrm{G}\ \rho(g)(\hat{e}_{nL}) = \rho(gg_L)(\hat{e}_n) = \rho(g_M h)(\hat{e}_n)$ since gg_L, as any element of G, necessarily belongs to a unique left coset g_MH, but $\rho(g_M h)(\hat{e}_n) = \rho(g_M)\rho(h)(\hat{e}_n) = \rho(g_M)\xi(h)(\hat{e}_n) = \rho(g_M)\theta(h)(\hat{e}_n) = \rho(g_M)\sum_m \Gamma_{\mathrm{H}mn}(\hat{e}_m) = \sum_m \Gamma_{\mathrm{H}mn}\rho(g_M)(\hat{e}_m) = \sum_m \Gamma_{\mathrm{H}mn}(\hat{e}_{mM})$. It follows that the matrix representation $\Gamma_{\mathrm{H}\uparrow\mathrm{G}}$ associated with the induced representation ρ with respect to the basis $\{\hat{e}_{nL}\}_{n=1,\ldots,d_{\mathrm{W}},L=1,\ldots,[\mathrm{G:H}]}$ is a matrix with the non zero-block component $\Gamma_{\mathrm{H}}((g_M)^{-1}gg_L)$ in the (L, M) entry iff $(g_M)^{-1}gg_L \in \mathrm{H}$ and zero-block components in the other entries:

$$\Gamma_{\mathrm{H}\uparrow\mathrm{G}}(g) = \sum_{h\in\mathrm{H}} \Xi(g,h) \otimes \Gamma_{\mathrm{H}}(h) \quad \text{with} \quad \Xi(g,h)_{M,L} = \delta(gg_L, g_M h) \tag{4.11}$$

where δ stands for the Kronecker symbol: $\delta(gg_L, g_M h) = 1$ iff $gg_L = g_M h$ and 0 otherwise. **$\boldsymbol{\Gamma_{\mathrm{H}\uparrow\mathrm{G}}}$ is said induced by $\boldsymbol{\Gamma_{\mathrm{H}}}$**. The notation with an arrow $\Gamma_{\mathrm{H}\uparrow\mathrm{G}}$ is often evocative. We for instance have the transitivity property: if $\mathrm{H} \subset \mathrm{G} \subset \mathrm{M}$, $\mathrm{W} \subset \mathrm{V} \subset \mathrm{U}$ and $\eta : \mathrm{M} \to \mathrm{GL}(\mathrm{U}, \mathbb{C})$ is induced by $\rho : \mathrm{G} \to \mathrm{GL}(\mathrm{V}, \mathbb{C})$, itself induced by $\theta : \mathrm{H} \to \mathrm{GL}(\mathrm{W}, \mathbb{C})$ then $\eta : \mathrm{M} \to \mathrm{GL}(\mathrm{U}, \mathbb{C})$ is induced by $\theta : \mathrm{H} \to \mathrm{GL}(\mathrm{W}, \mathbb{C})$. With arrows this reads more concisely: $\Gamma_{(\mathrm{H}\uparrow\mathrm{G})\uparrow\mathrm{M}} = \Gamma_{\mathrm{H}\uparrow\mathrm{M}}$.

The character $\chi_{\mathrm{H}\uparrow\mathrm{G}}$ of the representation ρ induced by θ is straightforwardly obtained as

$$\chi_{\mathrm{H}\uparrow\mathrm{G}}(g) = \mathrm{Tr}(\Gamma_{\mathrm{H}\uparrow\mathrm{G}}(g)) = \sum_{\substack{g_L\in\mathrm{R}\\ (g_L)^{-1}gg_L\in\mathrm{H}}} \chi_{\mathrm{H}}((g_L)^{-1}gg_L) = \frac{1}{n_{\mathrm{H}}} \sum_{\substack{f\in\mathrm{G}\\ f^{-1}gf\in\mathrm{H}}} \chi_{\mathrm{H}}(f^{-1}gf) \quad \forall g \in \mathrm{G} \tag{4.12}$$

where χ_{H} stands for the character of θ : $\chi_{\mathrm{H}}(h) = \mathrm{Tr}(\Gamma_{\mathrm{H}}(h))\ \forall h \in \mathrm{H}$. The square of the character $\chi_{\mathrm{H}\uparrow\mathrm{G}}$ is easily computed as:

$$\begin{aligned}
\left\langle \chi_{\mathrm{H}\uparrow\mathrm{G}} \mid \chi_{\mathrm{H}\uparrow\mathrm{G}} \right\rangle_{\mathrm{G}} &= \frac{1}{n_{\mathrm{G}}} \sum_{g\in\mathrm{G}} (\chi_{\mathrm{H}\uparrow\mathrm{G}}(g))^{\star} \chi_{\mathrm{H}\uparrow\mathrm{G}}(g) \\
&= \frac{1}{n_{\mathrm{G}}} \sum_{g\in\mathrm{G}} \sum_{g_L} (\chi_{\mathrm{H}}((g_L)^{-1}gg_L))^{\star} \chi_{\mathrm{H}}((g_L)^{-1}gg_L) \\
&+ \frac{1}{n_{\mathrm{G}}} \sum_{g\in\mathrm{G}} \sum_{g_M \neq g_L} (\chi_{\mathrm{H}}((g_L)^{-1}gg_L))^{\star} \chi_{\mathrm{H}}((g_M)^{-1}gg_M)
\end{aligned} \tag{4.13}$$

The first sum is simplified into $\frac{1}{n_{\mathrm{G}}}\,[\mathrm{G} : \mathrm{H}]\, n_{\mathrm{H}}\, \langle \chi_{\mathrm{H}} \mid \chi_{\mathrm{H}} \rangle_{\mathrm{H}} = \langle \chi_{\mathrm{H}} \mid \chi_{\mathrm{H}} \rangle_{\mathrm{H}}$. It follows that $\Gamma_{\mathrm{H}\uparrow\mathrm{G}}$ is irreducible iff Γ_{H} is irreducible and the second sum is null. If **H is a normal subgroup of G** then, defining the conjugate Γ^L_{H} of Γ_{H} by g_L as $\Gamma^L_{\mathrm{H}}(h) = \Gamma_{\mathrm{H}}((g_L)^{-1}hg_L)\ \forall h \in \mathrm{H}$, one shows that **the second sum is null iff none of the conjugate $\boldsymbol{\Gamma^L_{\mathrm{H}}}$ of $\boldsymbol{\Gamma_{\mathrm{H}}}$ by $\boldsymbol{g_L}$ has common irreducible component with another distinct conjugate $\boldsymbol{\Gamma^M_{\mathrm{H}}}$ of $\boldsymbol{\Gamma_{\mathrm{H}}}$ by $\boldsymbol{g_M}$.**

Let $\varphi_{\mathrm{H}} \in \mathbb{C}\,[\mathfrak{C}_{\mathrm{H}}]$ be any class function on H. The complex valued function $\varphi_{\mathrm{H}\uparrow\mathrm{G}} \in \mathbb{C}\,[\mathrm{G}]$ on G defined by the formula

$$\varphi_{\mathrm{H}\uparrow\mathrm{G}}(g) = \frac{1}{n_{\mathrm{H}}} \sum_{\substack{f\in\mathrm{G}\\ f^{-1}gf\in\mathrm{H}}} \varphi_{\mathrm{H}}(f^{-1}gf) \quad \forall g \in \mathrm{G} \tag{4.14}$$

is said induced by φ_{H}. Since it is a linear combination of characters, $\varphi_{\mathrm{H}\uparrow\mathrm{G}}$ is a class function: $\varphi_{\mathrm{H}\uparrow\mathrm{G}} \in \mathbb{C}\,[\mathfrak{C}_{\mathrm{G}}]$. Equation (4.14) merely extend the concept of induction to any class function. Consider reciprocally $\psi_{\mathrm{G}} \in \mathbb{C}\,[\mathfrak{C}_{\mathrm{G}}]$ and denote $\psi_{\mathrm{G}\downarrow\mathrm{H}}$ the restriction of ψ_{G} to the subgroup H of G. This allows formulating in a symmetric form the **Frobenius Reciprocity Theorem**:

$$\left\langle \varphi_{\mathrm{H}} \mid \psi_{\mathrm{G}\downarrow\mathrm{H}} \right\rangle_{\mathrm{H}} = \left\langle \varphi_{\mathrm{H}\uparrow\mathrm{G}} \mid \psi_{\mathrm{G}} \right\rangle_{\mathrm{G}} \quad \forall \varphi_{\mathrm{H}} \in \mathbb{C}\,[\mathfrak{C}_{\mathrm{H}}]\ \psi_{\mathrm{G}} \in \mathbb{C}\,[\mathfrak{C}_{\mathrm{G}}] \tag{4.15}$$

Equation (4.15) is useful is establishing the **Mackey's criterion** of irreducibility of the induced representations. Also required is the notion of double cosets: $\mathrm{H}g\mathrm{K} = \{hgk \mid h \in \mathrm{H},\ k \in \mathrm{K}\}$ for a pair (H,K) of subgroups of G. These partition the group G into equivalence classes. Let S be a set of representatives obtained with H = K, on choosing a single element in each distinct double coset.

A matrix representation Γ_{H}^{s} of the subgroup $\mathrm{H}_s = g_s \mathrm{H} g_s^{-1} \cap \mathrm{H}$ of H is defined for each g_s in S by putting $\Gamma_{\mathrm{H}}^{s}(h) = \Gamma_{\mathrm{H}}(g_s^{-1} h g_s)$ for $h \in \mathrm{H}$. One then shows that $\Gamma_{\mathrm{H}\uparrow\mathrm{G}}$ **is irreducible iff** Γ_{H} **is irreducible and** Γ_{H}^{s} **and** $\Gamma_{\mathrm{H}\downarrow\mathrm{H}^s}$ **are disjoints** $\forall g_s \notin \mathrm{H}$**, that is have no common irreducible component**. If H is a normal subgroup of G then $\mathrm{H}^s = \mathrm{H}$ and $\Gamma_{\mathrm{H}\uparrow\mathrm{G}}$ is irreducible iff Γ_{H} is irreducible and not equivalent to any of its conjugate Γ_{H}^{s} by $g_s \notin \mathrm{H}$.

The concept of induced representations provide powerful tools to demonstrate a variety of important theorems. We only mention among them the Artins'Theorem, which allows stating that each character of a group G is a linear combination with rational coefficients of characters of representations induced from cyclic subgroups of G. Induction is also extremely efficient in the determination of the irreducible representations from representations of its subgroups. Note finally that the notion of induced representations extends with the same definition to the compact groups G so long as H is a closed subgroup of finite index. With infinite index the notion may be defined through the Hilbert space of square integrable functions on the group.

4.4 Searching irreducibles

An essential problem of representation analysis is whether algorithmic procedures might be forged that would allow finding out the invariant subspaces of any linear representation and the invariant complements. A general method to determine the Character Table of any finite group can be given. In that purpose let us consider back the conjugacy classes of a group.

We may define the "product" of two conjugacy classes $\mathfrak{C}_i$ and $\mathfrak{C}_j$ formally as the set $\mathfrak{C}_i\mathfrak{C}_j = \{g_i g_j \mid g_i \in \mathfrak{C}_i,\ g_j \in \mathfrak{C}_j\}$. If $g \in \mathfrak{C}_i\mathfrak{C}_j$ then any conjugate to g is also the product of an element of $\mathfrak{C}_i$ by an element of $\mathfrak{C}_j$, merely because $hg_i g_j h^{-1} = hg_i h^{-1} hg_j h^{-1}$. In other words, if an element of the conjugacy class $\mathfrak{C}_l$ appears a given number $\mathrm{C}(\mathfrak{C}_i\mathfrak{C}_j\mathfrak{C}_l)$ of times in the set $\mathfrak{C}_i\mathfrak{C}_j$ then every other element of the same conjugacy class $\mathfrak{C}_l$ will appear the same number $\mathrm{C}(\mathfrak{C}_i\mathfrak{C}_j\mathfrak{C}_l)$ of times in the set $\mathfrak{C}_i\mathfrak{C}_j$. This means that the conjugacy class product $\mathfrak{C}_i\mathfrak{C}_j$ expands onto conjugacy classes $\mathfrak{C}_l$ as

$$\mathfrak{C}_i\mathfrak{C}_j = \sum_l \mathrm{C}(\mathfrak{C}_i\mathfrak{C}_j\mathfrak{C}_l)\,\mathfrak{C}_l \tag{4.16}$$

where the class multiplication coefficients are strictly positive integers: $\mathrm{C}(\mathfrak{C}_i\mathfrak{C}_j\mathfrak{C}_l) \in \mathbb{N} - \{0\}$. $\mathfrak{C}_i\mathfrak{C}_j = \mathfrak{C}_j\mathfrak{C}_i$, since $g_i g_j = g_j(g_j^{-1} g_i g_j)$, so that $\mathrm{C}(\mathfrak{C}_i\mathfrak{C}_j\mathfrak{C}_l) = \mathrm{C}(\mathfrak{C}_j\mathfrak{C}_i\mathfrak{C}_l)$. The expansion in the equation 4.16 contains the conjugacy class $\mathfrak{C}_l = \{e\}$, where e is the unit of the group G, iff the two conjugacy classes $\mathfrak{C}_i$ and $\mathfrak{C}_j$ are inverse of each, merely because $g_i g_j = e \Leftrightarrow g_i = g_j^{-1}$, and whenever this is so the conjugacy class e will appear $n_{\mathfrak{C}_i}$ times in the conjugacy class product of $\mathfrak{C}_i$ with itself if it is ambivalent and with its inverse if this is distinct from it. In other words,

$$\mathrm{C}(\mathfrak{C}_i\mathfrak{C}_j\{e\}) = \begin{cases} n_{\mathfrak{C}_i} \text{ if } \mathfrak{C}_j = \mathfrak{C}_i^{-1} \\ 0 \quad \text{otherwise} \end{cases} \tag{4.17}$$

Summing the linear operators $\rho_k(g)$ over a class $\mathfrak{C}_i$ the linear operator $\rho_i^k = \sum_{g_i \in \mathfrak{C}_i} \rho_k(g_i)$ is defined on the representation space V. ρ_i^k belongs to $\mathrm{End}_{\mathrm{G}}(V)$[21] so, by Schur 1, $\exists \lambda_i \in \mathbb{C} : \rho_i^k = \lambda_i 1_{\mathrm{V}_k}$ (cf. Section 2.8), which in terms of characters is transcribed into $n_{\mathfrak{C}_i}\chi_i^k = \lambda_i \chi^k(e)$. As from the

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$$\rho(h) \circ \rho_i^k \circ \rho(h)^{-1} = \sum_{g \in \mathfrak{C}_i} \rho(h) \circ \rho_k(g) \circ \rho(h)^{-1} = \sum_{g \in \mathfrak{C}_i} \rho_k(hgh^{-1}) = \sum_{u = hgh^{-1} \in \mathfrak{C}_i} \rho_k(u) = \rho_i^k.$$

equation 4.16 it is inferred that $\rho_i^k \circ \rho_j^k = \sum_{g_i \in \mathfrak{C}_i} \rho_k(g_i) \circ \sum_{g_j \in \mathfrak{C}_j} \rho_k(g_j) = \sum_{g_i \in \mathfrak{C}_i} \sum_{g_j \in \mathfrak{C}_j} \rho_k(g_i g_j) = \sum_l \mathrm{C}(\mathfrak{C}_i \mathfrak{C}_j \mathfrak{C}_l) \sum_{g_l \in \mathfrak{C}_l} \rho_k(g_l) = \sum_l \mathrm{C}(\mathfrak{C}_i \mathfrak{C}_j \mathfrak{C}_l)\, \rho_l^k$, so $\lambda_i 1_{\mathrm{V}_k} \lambda_j 1_{\mathrm{V}_k} = \sum_l \mathrm{C}(\mathfrak{C}_i \mathfrak{C}_j \mathfrak{C}_l)\, \lambda_l 1_{\mathrm{V}_k}$ whence

$$(n_{\mathfrak{C}_i} \chi_i^k)\,(n_{\mathfrak{C}_j} \chi_j^k) = \chi^k(e) \sum_l \mathrm{C}(\mathfrak{C}_i \mathfrak{C}_j \mathfrak{C}_l)\,(n_{\mathfrak{C}_l} \chi_l^k) \tag{4.18}$$

If $\mathcal{N}_{\mathfrak{C}}$ is the number of the conjugacy classes of the group G then this makes up a system of $\mathcal{N}_{\mathfrak{C}}^2$ equations over the $\mathcal{N}_{\mathfrak{C}}$ variables χ_i^k $(i = 1, \mathcal{N}_{\mathfrak{C}})$. This is the starting point of a variety of algorithms to determine the Character Tables of the finite groups. Consult [6] for further details. The computations of irreducible representations are harder, as emphasized in [7].

Arithmetic properties of the characters are also extremely useful. Note that since every element of a finite group has finite order, the character values always are sums of eigenvalues that are roots of the multiplicative unit, that is to say roots of a polynomial with coefficients in the set of integers $\mathbb{Z}$. This defines algebraic integers. It then follows, for instance, from the equation (3.13) that the dimensions d_k of the irreducible representations $\rho_k : \mathrm{G} \to \mathrm{GL}(\mathrm{V}_k, \mathbb{C})$ are all divisors of the order n_{G} of the group G, since the set of algebraic integers is closed under addition and multiplication and since algebraic integers given as rationals are in fact integers.

4.5 Group actions

Let $\rho : \mathrm{G} \to \mathrm{Aut}(\mathrm{X})$ be a **representation of a group G on a mathematical object X**. One always may define a function $\alpha : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ that canonically maps each couple $(g, x) \in \mathrm{G} \times \mathrm{X}$ into $\alpha(g, x) = \rho(g)(x) \in \mathrm{X}$. It is straightforward to show that α preserves the law of G, namely $\alpha(gh, x) = \alpha(g, \alpha(h, x))\ \forall g, h \in \mathrm{G}\ \forall x \in \mathrm{X}$, since ρ is an homomorphism, and that the unit e of G is neutral for α, namely $\alpha(e, x) = x\ \forall x \in \mathrm{X}$, because $\rho(e)$ necessarily is the identity of Aut(X). In other words, α is nothing but an **action of the group G on the mathematical object X**. Conversely, given an action $\alpha : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ one always may define a function $\rho : \mathrm{G} \to \mathrm{Aut}(\mathrm{X})$ that canonically maps each $g \in \mathrm{G}$ into the isomorphism $\rho(g) : x \mapsto \alpha(g, x)$ of X. It is not more difficult to demonstrate that the properties of an action imply that ρ is a group homorphism. Accordingly, it is equivalent to define a representation ρ of a group G on a mathematical object X or an action α of this group G on that object X. It then is tempting to state that a representation is identical to an action, but that would make up a mathematical abuse.

Using either of the two concepts of action or of representation, symmetry can be defined in a very wide context. A subset Y of X is said **invariant** under a subgroup S of G if $\{\alpha(g, x) \mid (g, x) \in \mathrm{S} \times \mathrm{Y}\} \subseteq \mathrm{Y}$. The elements of S then are called the **symmetries** of Y.

A group action $\alpha : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ is said **isomorphic** to a group action $\beta : \mathrm{G} \times \mathrm{Y} \to \mathrm{Y}$, symbolically $\boldsymbol{\alpha \sim \beta}$, if they are intertwined with an isomorphism, namely if there exists an isomorphism $\theta : \mathrm{X} \to \mathrm{Y}$ which is equivariant: $\theta(\alpha(g, x)) = \beta(g, \theta(x))\ \forall (g, x) \in \mathrm{G} \times \mathrm{X}$. Of course, if $\rho : \mathrm{G} \to \mathrm{Aut}(\mathrm{X})$ and $\xi : \mathrm{G} \to \mathrm{Aut}(\mathrm{Y})$ are the representations canonically associated with α and β then $\theta \circ \rho(g) = \xi(g) \circ \theta\ \forall g \in \mathrm{G}$ that is $\boldsymbol{\rho \sim \xi}$.

The set $\mathrm{Orb}_\alpha(x) = \{\alpha(g, x) \mid g \in \mathrm{G}\}$ by definition is the **orbit** of $x \in \mathrm{X}$. Writing $x \mathcal{R}_\alpha y$ for $y \in \mathrm{Orb}_\alpha(x)$ one gets an equivalence relation, which partition the set X into orbits. The quotient set defines the **orbit space** X | G. If $\alpha : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ is an action of a finite group G on a manifold then X | G is an orbifold with the singularities on the fixed points of α in X. Interest in the orbifolds strongly raised in the context of the geometrization conjecture, formulated by Thurston then proved by Perelman, as essential pieces of manifold decompositions. An **action $\boldsymbol{\alpha}$ is transitive** if $\mathrm{Orb}_\alpha(x) = \mathrm{X}$.

The set $\mathrm{Stab}_\alpha(x) = \{g \in \mathrm{G} \mid \alpha(g, x) = x\}$ by definition is the **stabilizer** of $x \in \mathrm{X}$. It forms a subgroup of G, whatever x in X. It is also called a **little group**. One easily establishes that

$\mathrm{Stab}_\alpha(\alpha(g,x)) = g\,\mathrm{Stab}_\alpha(x)g^{-1}$.[22] It follows that the collection $\{\mathrm{Stab}_\alpha(\alpha(g,x)) \mid g \in \mathrm{G}\}$ of the stabilizers of the elements of an orbit $\mathrm{Orb}_\alpha(x)$ forms a conjugacy class of subgroups of G. If $\mathrm{Stab}_\alpha(x) = G$ then $\mathrm{Orb}_\alpha(x) = x$ and x is termed a **fixed point**. If $\mathrm{Stab}_\alpha(x) = \{e\}$ then $\mathrm{Orb}_\alpha(x)$ is termed a **principal orbit**. An **action α is effective** if all its orbits are principal: $\mathrm{Stab}_\alpha(x) = \{e\}\ \forall x \in \mathrm{X}$, which means that every element of G other than the unit e of G acts by changing every element of X.

The function $\alpha_x : \mathrm{G}/\mathrm{Stab}_\alpha(x) \to \mathrm{Orb}_\alpha(x)$, from the set of the left cosets of the stabilizer $\mathrm{Stab}_\alpha(x)$ in G to the orbit $\mathrm{Orb}_\alpha(x)$ is well defined and bijective. It then is inferred that: i- if G is finite then the number of elements of any orbit with the same conjugacy class of stabilizers as $\mathrm{Orb}_\alpha(x)$ is $n_{\mathrm{Orb}_\alpha(x)} = n_{\mathrm{G}}/n_{\mathrm{Stab}_\alpha(x)}$, denoting n_{E} the number of elements in a set E. ii- if α is an infinitely differentiable action of a Lie group then any orbit with the same conjugacy class of stabilizers as the orbit $\mathrm{Orb}_\alpha(x)$ is a manifold of dimension $d_{\mathrm{Orb}_\alpha(x)} = d_{\mathrm{G}} - d_{\mathrm{Stab}_\alpha(x)}$. If $d_{\mathrm{Orb}_\alpha(x)} = d_{\mathrm{G}} - d_{\mathrm{Stab}_\alpha(x)} = 0$ then the orbit is finite and its cardinal is the quotient of the number of connected components of G over the number of connected components of S.

A **stratum** by definition is the union of the orbits with the same conjugacy class of stabilizers. An example is the set of the fixed points of the action. Another is the union of the principal orbits, which consists in the points that are changed under any element of G other than the unit e of G. If α is an infinitely differentiable action of a compact group G on a real manifold X then every real valued function invariant with respect to G possesses extrema on each stratum corresponding to maximal little groups, namely proper little group not contained in any other proper little group, and all real valued function invariant with respect to G have in common orbits of extrema, which precisely are those critical in their stratum (consult [8]).

5. CONCLUSION

It is hoped that this little trip to the mathematical lands of linear representations of groups was not boring in spite of the many digression made with respect to the initial scope of the lecture and that, instead, was rather pleasant and enjoyable by providing an abstract glimpse of the basics on which the theory is founded. The reported literature provides more details. Clearly, it by no way is exhaustive and emanates only from the author's own arbitrary taste.

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[22] Indeed, $h \in \mathrm{Stab}_\alpha(\alpha(g,x)) \Leftrightarrow \alpha(h,\alpha(g,x)) = \alpha(g,x) \Rightarrow \alpha(g^{-1}hg,x) = \alpha(g^{-1},\alpha(h,\alpha(g,x))) = \alpha(g^{-1}g,x) = x \Rightarrow \exists f \in \mathrm{Stab}_\alpha(x)$ s.t. $h = gfg^{-1}$. In addition, $h = gfg^{-1} = h' = gf'g^{-1} \Leftrightarrow f = f'$.